Two puzzles concerning measures of uncertainty and the positive Boolean connectives

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Abstract. The two puzzles are the *Lottery Paradox* and the *Amalgamation Paradox*, which both point out difficulties for aggregating uncertain information. A generalization of the lottery paradox is presented and a new form of an amalgamation reversal is introduced. Together these puzzles highlight a difficulty for introducing measures of uncertainty to a variety of logical knowledge representation frameworks. The point is illustrated by contrasting the constraints on solutions to each puzzle with the structural properties of the preferential semantics for non-monotonic logics (System P), and also with systems of normal modal logics. The difficulties illustrate several points of tensions between the aggregation of uncertain information and aggregation according to the monotonically positive Boolean connectives, $\wedge$ and $\vee$.

1 Introduction

Uncertainty is a fundamental and unavoidable feature of our relationship to the world, so it is important to know how to represent uncertainty and how to reason about it. Within AI we see various attempts to combine measures of uncertainty with logical calculi in multi-agent systems [11], robotics [34], logic programming [19], security and verification [14], causal and non-monotonic reasoning [30, 17] among other fields. But measure functions behave very differently than logical truth functions, and combining both into one framework is a surprisingly subtle undertaking. This paper presents a pair of puzzles designed to highlight some rudimentary difficulties for reconciling aggregation according to combinations of probabilities with aggregation according to the monotonically positive Boolean connectives, $\wedge$ and $\vee$. The two puzzles are Henry Kyburg's *lottery paradox* [21] and the *amalgamation paradox* [13].

This essay presents a generalized version of the lottery paradox, and introduces a new type of amalgamation reversal. The generalized version of the lottery paradox is important because this form of the problem subverts several recent strategies for avoiding it. This point is discussed briefly here in connection with the treatment of *conjunction* and the rule of adjunction within probabilistic logic [35], within cumulative non-monotonic logics [26, 17], and within normal modal logics [4, 24]. This generalized form of the lottery paradox is discussed in more detail in [36].

The importance of the new amalgamation reversal is that this subverts the classic experimental design strategies introduce by I. J. Good and Y. Mittal [13] for avoiding reversal effects. The reason is that the classification categories of the new version are not (necessarily) exclusive, which reflects the behavior of Boolean disjunction in cumulative non-monotonic logics, and also reflects the behavior of Boolean disjunction within *classical* modal logics [27, 32, 9]. This point is discussed briefly in [24]. The aim of this essay is to discuss this issue in more detail. By highlighting recent work on the lottery paradox, together with this new type of amalgamation reversal, the essay offers a note of caution to applied logicians concerning attempts to combine measure theory and logic for the purpose of representing cogent reasoning under conditions of uncertainty.

The essay is organized as follows. Section 2 presents the original lottery paradox and a generalized variant. Section 3 presents an example of a standard amalgamation reversal. Following Good and Mittal, a generalized form of the puzzle is presented in terms of two-by-two contingency tables, and the example is represented in these terms. Representing examples in these terms helps to illustrate the scope of the amalgamation paradox, since a variety of measures can induce a reversal. We also mention Good and Mittal's analytical results that they use to specify necessary and sufficient conditions for avoiding reversal effects. In section 4 sub-structural conditions for cumulative non-monotonic logics are presented, and the properties for conjunction [**And**] and disjunction [**Or**] are compared with the lottery paradox, and with the amalgamation paradox, respectively.

2 The Lottery Paradox

Henry Kyburg's *lottery paradox* [21, p. 197] arises from considering a fair 1000 ticket lottery that has exactly one winning ticket. If this much is known about the execution of the lottery it is therefore rational to accept that one ticket will win. Suppose that an event is very likely if the probability of its occurring is greater than 0.99. On these grounds it is presumed rational to accept the proposition that ticket 1 of the lottery will not win. Since the lottery is fair, it is rational to accept that ticket 2 won't win either—indeed, it is rational to accept for any individual ticket i of the lottery that ticket i will not win. However, accepting that ticket 1 won't win, accepting that ticket 2 won't win, . . . , and accepting that ticket 1000 won't win entails that it is rational to accept that no ticket will win, which entails that it is rational to accept the contradictory proposition that one ticket wins and no ticket wins.

The lottery paradox was designed to demonstrate that three attractive principles governing *rational acceptance* lead to contradiction, namely that

1. It is rational to accept a proposition that is very likely true,
2. It is not rational to accept a proposition that you are aware is inconsistent, and

3. If it is rational to accept a proposition A and it is rational to accept another proposition A', then it is rational to accept $A \wedge A'$,

are jointly inconsistent.

The lottery paradox has sparked an enormous literature [36, see bibiography]. Kyburg's own view was to accept the first two principles and reject the third aggregation principle, and his innovative theory of probability is based upon these commitments [21, 22]. Orthodox Bayesians tend to resolve the paradox by accepting the second and third principles and rejecting the first, and with it the notion of rational acceptance underpinning the Kyburgian approach to probability [18, 8]. Thus, in the 1960s and 1970s, discussion of the 'lottery paradox' was associated with debates about the merits of different interpretations of probability and various frameworks for representing uncertainty. Since then epistemologists have tended to follow the orthodox line, which is why the paradox is linked to discussions of skepticism, and conditions for (reasonably) asserting knowledge claims. And AI researchers, economists, and mathematical psychologists have followed this track, too, often drawn by obvious mathematical advantages. But it is far less clear that these formal advantages translate to accurate representations of uncertainty.

Philosophical logicians, on the other hand, have tended to look for ways to weaken one or more of the governing principles for rational acceptance, and there are several approaches one may consider, including Jim Hawthorne and Luc Bovens's logic of belief [16], Bryson Brown's application of preservationist paraconsistent logic [7], Horacio Arló-Costa's appeal to classical modalities [4], Joe Halpern's use of first-order probability [15], Igor Douven and Timothy Williamson's appeal to cumulative consequence relations [10], and this author's use of 1-monotone capacities [35].

The generalization of the lottery paradox follows from pointing out some inessential features of the original thought experiment. For instance, the fact that one and only one ticket will be drawn is inessential to the paradox.[1] Furthermore, the puzzle is not necessarily avoided by turning to decision theory. The force of the lottery paradox is the relationship between uncertain evidence and reasonable belief, rather than the relationship between uncertain belief and rational action. We can illustrate this point with the following generalization of the puzzle.

When I book a flight from Philadelphia to Denver I judge it rational to accept that I will arrive without serious incident since I stand rough odds of $1 : 350,000$ of dying on a plane trip in the U.S. any given year,[2] and stand considerably better odds than this of arriving without incident if I book passage on a regularly scheduled U.S. commercial flight. I judge these odds to be those that characterize my flight from Philadelphia to Denver. Nevertheless, I believe that there will be a fatality from an airline accident on a U.S. carrier in the

[1] Compare to [16].
[2] According to the *National Safety Council*

coming year even though I don't believe that there must be at least one fatal accident each year. Indeed, 2002 marked such an exception.[3]

Notice that I am not speaking of my decision to board the plane, but my belief that I will get off the flight alive. The reason why I believe I will land safely in Denver is that I stand odds better than $1 : 350,000$ of surviving and, as far as I know, it is reasonable to view my trip as a random member of the class of domestic trips on US domestic carriers. My belief that I will land safely is distinct from what actions I may be willing to take given this stance, such as booking a car or hotel, or even the act of getting on that flight. In short, I believe that I will survive this flight to Denver. My belief that I will survive the trip is part of my background knowledge I have to base my decisions about actions on options I will, or should, take while in Denver. My belief that there will be an accidental fatality on US carriers in this coming year is unlikely to serve as a basis for actions I might take, but it is a belief that I nonetheless have. This belief is revealed to me by my surprise, which perhaps the reader shares, when I learned that there was a recent year in which there were no fatalities on US airline carriers.

We think that the notion of accepting a claim on the basis of uncertain but compelling evidence is fundamental [24]. But if one holds that a knowledge representation framework should accommodate a representation of full but uncertain belief, this leads to structural difficulties. We shall return to the consequences from adopting this view in section 4. But first let's introduce our other puzzle, the amalgamation paradox.

3 Amalgamation Paradox

Imagine that your university wishes to increase the number of women on faculty. To achieve this goal, all hiring departments are instructed to discriminate in favor of women. Suppose that the university advertises positions in the Department of Mathematics and in the Department of Sociology, and only those departments. Five men apply for the openings in Mathematics and one is hired, while eight women apply and two are hired. Therefore the success rate for men is twenty percent, and the success rate for women is twenty-five percent. Thus, the Mathematics Department is in compliance with the University policy. In the Sociology Department eight men apply and six are hired, and five women apply and four are hired. Therefore the success rate for men is seventy-five percent and for women it is eighty percent. Thus, the Sociology Department is also in compliance with University policy. An equal number of female and male candidates applied for jobs, 13, but overall 7 men and 6 women are hired. Thus the success rate for male applicants is approximately 54% but the success rate for female

[3] According to the 2002 *National Transportation Safety Board* there were 34 accidents on U.S. commercial airlines during 2002 but zero fatalities, a first in twenty years.

applicants is approximately 46%. Thus, the university is *not* in compliance with its new hiring policy.[4]

Table 1. University Recruitment Example

	Men		**Women**
Mathematics	$\frac{1}{5}$	$<$	$\frac{2}{8}$
Sociology	$\frac{6}{8}$	$<$	$\frac{4}{5}$
University	$\frac{7}{13}$	$>$	$\frac{6}{13}$

The amalgamation paradox is I.J. Good and Y. Mittal's generalization [13] of a puzzle first noticed by Pearson [31, 277-8] and also by Yule [38], which is sometimes referred to as 'Simpson's paradox' or the 'reversal paradox'. Yule pointed out that a pair of attributes may not necessarily exhibit a relationship within a population at large even when it is observed in every subpopulation, which is precisely what is illustrated by the University example, while Pearson stressed an analogous point about correlation measures for continuous (i.e., non-categorical) data.

Following Good and Mittal, the paradox can be expressed in terms of a two-by-two contingency table, such as $\mathbf{t} = [a, b; c, d]$, where the entries a, b, c, d sum to N, the sample size, $abcd \neq 0$, and N is assumed large enough to ignore sampling variation.

A measure of association m of $\mathbf{t}$, $m(\mathbf{t})$, is a function of a, b, c, d to $\{1, 0\}$. So, $m(\mathbf{t})$ is either 1 or 0, either 'true' or 'false'. Amalgamation is then defined for mutually exclusive sets of two-by-two contingency tables.

Definition 1 (Amalgamation). *Suppose* $\mathbf{t_i} = [a_i, b_i; c_i, d_i]$*, for* $i = 1, \ldots, n$*, is a set of size* n *of mutually exclusive two-by-two contingency tables. Amalgamation is a single table* $\mathbf{T}$ *composed of the sums of each coordinate in the tables, elements of* $\mathbf{T} = [A, B; C, D] = [\sum_i^n a_i, \sum_i^n b_i;\ \sum_i^n c_i, \sum_i^n d_i]$.

Note that $A + B + C + D = N$. Let $a_i + b_i + c_i + d_i = N_i$; for simplicity, assume that N_i is proportional to the fraction p_i of the population that makes up the ith subpopulation.[5]

[4] This example is based upon a sex discrimination lawsuit that was brought against the University of California, Berkeley, and discussed in [6].

[5] This is not always a reasonable assumption. We might want to sample people from Portugal and people from India, but it would be unreasonable to insist upon taking sample sizes that were proportional to each population. If the sample sizes are sufficiently large, however, we can scale the tables to force N_i to be proportional to p_i. And this scaling is reasonable if sample sizes are large enough to ignore sampling variation, which we assume.

Then the Amalgamation paradox occurs if the maximum measure of association among a set of contingency tables is less than the measure of association of the amalgamation of those tables, or the measure of association for the amalgamation of a set of contingency tables is less than the minimal measure of association among a set of contingency tables, that is if

$$\max_i m(\mathbf{t_i}) < m(\mathbf{A}) \quad \text{or} \quad m(\mathbf{A}) < \min_i m(\mathbf{t_i}).$$

In words, but of slightly stronger form, it is possible for *every* subpopulation to have an association measure pointing in one direction (i.e., either all equal to 0 or all equal to 1), but for the population itself to record a 'reverse' association measure (i.e., a measure of 1 or of 0, respectively).

Example 1 (University Recruitment Example). Let $\mathbf{t_i} = [a_i, b_i; c_i, d_i]$ for $i = 1, 2$ where the '1' is the table for the subpopluation of Mathematics Department recruitment data, '2' the subpopulation of Sociology Department recruitment data, a_i represents the number of males hired by department i, b_i the number of males not hired by i, c_i the number of females hired by i, and c_i the number of females not hired by i. $\mathbf{T} = [A, B; C, D]$ is the amalgamation of Mathematics and Sociology recruitment data, representing the University recruitment statistics. Thus,

(i) $\mathbf{t_1} = [1, 4; 2, 6]$; $\mathbf{t_2} = [6, 2; 4, 1]$; $\mathbf{T} = [7, 6; 6, 7]$;
(ii) $N_1 = N_2 = 13$; $N = 26$;
(iii) $m(\mathbf{t}) = \begin{cases} 1 & \text{if } \dfrac{a}{a+b} < \dfrac{c}{c+d}; \\ 0 & \text{otherwise.} \end{cases}$, and
(iv) $m(\mathbf{t_1}) = m(\mathbf{t_2}) = 1$, but $m(\mathbf{T}) = 0$.
(v) Therefore, $m(\mathbf{T}) < \min_i m(\mathbf{t_i})$, for all $i = 1, 2$. □

The advantage to representing the amalgamation paradox in terms of contingency tables is that one may then consider a variety of measures m that induce reversals. Good and Mittal do just this in [13], where they consider a (non-exhaustive) list of measures. They also define two conditions, *row-uniformity*, for some λ and $i = 1, \ldots, n$:

$$\frac{a_i + b_i}{c_i + d_i} = \lambda,$$

and *column-uniformity*, for some μ and $i = 1, \ldots$, n:

$$\frac{a_i + c_i}{b_i + d_i} = \mu.$$

Whether or not these conditions can be satisfied depends upon the geometry of the tables *and* the sampling procedures used in their construction, and Good and Mittal discuss some situations under which these conditions can be built into the design of an experiment and, hence, avoid reversal effects for the amalgamated table $\mathbf{T}$.

But Good and Mittal's results also depend upon the mutual exclusivity of each of the n contingency tables,$\mathbf{t_i} = [a_i, b_i; c_i, d_i]$. The form of reversal that we presented in [24] and which is discussed in more detail in the next section trades on the mutual-exclusivity of categories. Unlike Pearson's version, our example concerns categorical data. Unlike Yule's version, which is the form generalized by Good and Mittal, the categories fail to be mutually exclusive. Determining whether or not contingency tables satisfy the mutual exclusivity condition may not appear to be problematic from a experimental point of view, since (typically) enough is known about classification categories and experimental design to warrant assumptions that cells are mutually exclusive. However, from a logical point of view, we typically *do not* have this information available when considering the aggregation of information. Indeed, that is the point behind a logical connective. Since the truth conditions for Boolean disjunction are open with respect to items satisfying one or both categories, this means that there is no guarantee within the logical representation that the categories are mutually exclusive. We will return to this point in the next section, when we consider the 'Rotten Apple' example.

4 System P and Systems of Modal Logic

There is a long history behind providing probabilistic semantics for logical calculi [1, 33], which was taken up in AI [28, 25] in terms of providing a probabilistic semantics for satisfying the axioms of System P for non-monotonic conditionals, an axiom system first discussed in [20]. System P consists of a number of axioms and rules of inference that are taken by many[6] to be a conservative core any nonmonotonic system should contain.

Let $\mid\!\sim$ be a nonmonotonic consequence relation. Let $\vee, \wedge, \rightarrow$ and $\leftrightarrow$ be standard Boolean connectives in a classical propositional logic, $\rightarrow$ being the truth functional conditional. Let $\models \alpha$ denote α is valid. Finally, $\models \ulcorner \alpha \rightarrow \beta \urcorner$ can be equivalently expressed as $\alpha \models \beta$. The axioms of System P are:

$$\alpha \mid\!\sim \alpha \qquad \textbf{[Reflexivity]}$$

$$\frac{\models\ \alpha \leftrightarrow \beta; \quad \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma} \qquad \textbf{[Left Logical Equivalence]}$$

$$\frac{\models\ \alpha \rightarrow \beta; \quad \gamma \mid\!\sim \alpha}{\gamma \mid\!\sim \beta} \qquad \textbf{[Right Weakening]}$$

$$\frac{\alpha \mid\!\sim \beta; \quad \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \wedge \gamma} \qquad \textbf{[And]}$$

$$\frac{\alpha \mid\!\sim \gamma; \quad \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma} \qquad \textbf{[Or]}$$

$$\frac{\alpha \mid\!\sim \beta; \quad \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma} \qquad \textbf{[Cautious Monotonicity]}$$

[6] For example, see [26, 15, 10].

Both Ernest Adams [1, 2] and Judea Pearl [28, 29] have sought to make a connection between high probability and logic, but both have taken probabilities *arbitrarily* close to one as corresponding to knowledge. Although Pearl bases his approach on infinitesimal probabilities and Lehmann and Magidor base theirs on a non-standard probability calculus, each shares the view that 'acceptance' or 'full belief' is to be identified with maximal probability, a view that has been developed by [12] [3], defended in [15] and studied by [5], where the latter includes an important limitative result.

But this conception of 'full-belief' conflicts with the notion of rational acceptance that drives Kyburg's lottery paradox. Suppose that $\alpha \mid\!\sim \beta$ denotes that together with the suppressed background knowledge, α is good, but not necessarily certain, evidence for β. The idea then is that rational acceptance of β is determined relative to a fixed threshold for acceptance. But the lower bound of the probability of the conjunction $\beta \wedge \gamma$ is not higher than and can be lower than the smaller of the two lower bounds of the probabilities of β and of γ. Indeed, **[And]** is simply the rule of adjunction, the third purported principle for governing rational acceptance, and this principle is in direct conflict with full but uncertain belief. Some studies of sub-P, weakly aggregative systems include [35, 24, 17, 4].

Turning to modal logic, note that the schema $(\Box\phi \wedge \Box\psi) \leftrightarrow \Box(\phi \wedge \psi)$ is valid in *all* classes of normal modal logics. So, if we interpret $\Box$ to be a threshold-valued belief operator, we then see that the left-to-right direction,

(C) $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$,

is the adjunction rule. It is worth mentioning that (**C**) is *not* valid in all classes of *classical* modal logics [27, 32, 9], and the Scott-Montague semantics have been explored to characterize acceptance principles that restrict adjunction [4, 23].

Up to this point we have reviewed how the aggregation of probabilities clashes with the treatment of aggregation in systems of cumulative non-monotonic logic and with the distribution properties of Kripke structures. We have also remarked on the general reaction among applied logicians to the incompatibility between rational acceptance and the rule of adjunction [37]. There is a tendency to view attractive formal properties of logical systems to also be normative constraints of whatever the logic is imagined to represent. In this case, one finds the view that the clash between rational acceptance and the rule of adjunction is so much the worse for the principle of rational acceptance.

But the structural dissimilarities between probability and logical calculi are not restricted to a conflict over **[And]**—and, by extension, over **[Cautious Monotonicity]**. The rule **[Or]** is problematic as well in that it is susceptible to a form of amalgamation reversal that exploits the fact that Boolean disjunction does not guarantee that categories are mutually exclusive. Furthermore, the problematic behavior appears not just in normal modal logics but also in the more general class of classical modal logics.

Consider first a counter-example to **[Or]**, the Rotten Apple example.

Example 2 (Rotten Apple Example). Suppose the probability that a Jonathan apple (α) is not rotten (γ) is at least 90%, and the probability that an Ohio apple (β) is not rotten is at least 90%. Actually all non-rotten apples are Jonathan apples *and* from Ohio. In this case the lower bound of the probability that an apple is not rotten given that it is a Jonathan *or* from Ohio ($\alpha \vee \beta$) is 81.8%.

A joint probability distribution for this counter-example is as follows.

$\alpha\ \beta\ \gamma$	Probability	$\alpha\ \beta\ \gamma$	Probability
0 0 0	0	1 0 0	1/11
0 0 1	0	1 0 1	0
0 1 0	1/11	1 1 0	0
0 1 1	0	1 1 1	9/11

Given a lower bound requirement of 90%, we have $\alpha \mathrel{|\!\!\sim} \gamma$ ($\frac{9/11}{10/11}$) and $\beta \mathrel{|\!\!\sim} \gamma$ ($\frac{9/11}{10/11}$) but not $\alpha \vee \beta \mathrel{|\!\!\sim} \gamma$ ($\frac{9/11}{11/11}$). □

Turning to modal logic, observe that the range of modal systems captured by the Scott-Montague neighborhood semantics—i.e., classical, monotone, regular, normal—is not sufficient to mitigate against the type of reversal effect demonstrated in Example 2. The reason is that α and β here are non-tautological sentences, and facts about deducibility of classical modal logics ($\vdash_C$) entail that if $\alpha \vdash_C \gamma$ and $\beta \vdash_C \gamma$, then $\alpha \vee \beta \vdash_C \gamma$: deducibility is *monotonic*, and since both $\alpha \vdash_C \gamma$ and $\beta \vdash_C \gamma$ and $\alpha \subset \{\alpha \cup \beta\} \supset \beta$, then $\{\alpha \cup \beta\} \vdash_C \gamma$.

Finally, a word about 'exclusive or'. One might consider resolving the conflict with System P by replacing $\vee$ with a connective $\circ$, where $p \circ q$ is true if and only if $(p \vee q) \wedge \neg(p \wedge q)$. Notice, however, that $\circ$ is *not* a suitable logical connective for ensuring, compositionally, that at most one of its arguments is true: $p \circ (q \circ r)$ is true when p, q and r are all true.

5 Conclusion

Most of the warnings about combining probability and logic have concentrated upon the non-compositional character of calculating compound events, on the one hand, and upon the difference between how probabilities and extensional truth assignments are aggregated. The lottery paradox has long been viewed to encapsulate the crux of the problem. However, the introduction of this amalgamation reversal presents a new difficulty to efforts to combine logic and probability, and one that appears to have consequences of wider reach. The class of sub-P non-monotonic systems charted by [17] are predicated upon the **[Or]** axiom holding, for instance, and we see that the counter-example applies beyond the class of normal logics to include *all* systems of classical modal logics.[7]

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