AGM Revision in Classical Modal Logics

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Abstract

AGM-style revision operators are defined for several systems of classical modal logic.

Within AGM, consistency maintenance is done by classical logic. But the reliance on classical consistency presents a problem for exporting AGM revision to non-classical logics in general, and to modal logic in particular.

A general technique for solving this problem is to translate a non-classical logic into classical logic, together with a specification of consistency particular to the non-classical logic, perform the operation of revision on this translation within classical logic, then translate the result back into the non-classical logic we started with (Gabbay, Rodrigues, and Russo 2008). In the case of normal modal logic, both the modal language and the semantic structure must be translated into first-order logic, and this translation for well-known normal systems will rely upon well-known frame-theoretic properties, expressed in first-order logic, of modally defined classes of Kripke frames (Goldblatt 1993).

But there is a problem extending this technique to classical modal logics (Chellas 1980), because there is no (direct) correspondence between neighborhood frames and first-order logic. This paper proposes to solve this problem by adapting Marc Pauly's (Hansen 2003) technique of first simulating neighborhood structures by polymodal Kripke structures, then define a correspondence to first-order logic from the polymodal Kripke semantics wherein AGM revision can be defined.

In modal logic the technique of simulation was first used to construct counter-examples within polymodal modal logic to export back to monomodal systems of interest (Thomason 1974; 1975). More recently the technique has been used to study the relationship between neighborhood semantics and Kripke's relational semantics, with a particular focus on supplemented neighborhood models (Gasquet and Herzig 1996; Kracht and Wolter 1999; Hansen 2003). Supplemented neighborhood models underpin a variety of non-additive, monotone modal logics appearing in knowledge representation formalisms, including *Game Logic* (Parikh 1985), *Concurrent Propositional Dynamic Logic* (Goldblatt 1992), *Alternating-time logic* (Alur, Henzinger, and Kupferman 1992), *Risky Knowledge* (Kyburg and Teng 2002), and *Coalition Logic* (Pauly 2002). Non-monotone classical modal logics have also been pressed into service, including *Local Reasoning* (Fagin and Halpern 1988), and the logic of *Only Knowing* (Humberstone 1987; Levesque 1990; Halpern and Lakemeyer 1995), which are based on the *Inaccessible Worlds* semantics of (Humberstone 1983).

In addition to discovering properties of a logic by studying its simulation within a well-understood system, the techniques of simulation theory together with correspondence theory may be used to bring new capabilities to a classical modal logic. AGM belief revision is but one example, which is the subject of this paper.

Recent work in modal belief revision focuses on specifically tailored classical modal logics, namely dynamic epistemic logic (van Benthem 2007; van Eijck 2009), and polymodal normal logics, such as branching time temporal logic (Bonanno 2009). Each addresses particular issues that arise—principally updating and common knowledge, in the case of the former, and the interaction of temporal and epistemic operators on the standard interpretation in the latter. But, there are many interpretations of classical modal logics within knowledge representation other than these two, and one might like instead to see a general strategy for supplying AGM revision to systems of classical modal logic. The results of this paper supply that strategy.

Classical Modal Logic

To begin, we highlight the difference between neighborhood structures and standard Kripke structures. Whereas Kripke frames are characterized by a binary accessibility relation defined over a set of worlds, a **neighborhood frame** for the propositional modal language $\mathcal{L}_\nabla(\Phi)$ is a pair $\mathbb{F} = (W, \mathcal{N})$ where

a) W is a non-empty set of worlds,

b) $\mathcal{N} : W \mapsto \wp(\wp(W))$ is a neighborhood function, i.e. $\mathcal{N}(w) \subseteq \wp W$, for each $w \in W$.

If $\mathbb{F} = (W, \mathcal{N})$ is a neighborhood frame, Φ a countable set of propositional variables, and $V : \Phi \mapsto \wp(W)$ is a valuation on $\mathbb{F}$, then $\mathbb{M} = (W, \mathcal{N}, V)$ is a **neighborhood model** based on $\mathbb{F}$.

The satisfiability conditions for non-modal propositional formulas on neighborhood models are analogous to Kripke models, but modal necessity ($\nabla\varphi$) and possibility ($\Delta\varphi$) statements on neighborhood models are different. Like the normal modal logic (K) and its extensions, classical modal logics are based on the classical system (E) and the meaning of necessity statements in different classical systems is determined by the properties of neighborhood frames just as the meaning of necessity statements in different normal systems is determined by the properties of a Kripke frame. That said, there are four important classes of neighborhood models (minimal, supplemented, quasi-filters, augmented) that determine four modal systems (classical, monotone, regular, normal). The differences between these models can be reflected by the **truth conditions** for ($\nabla\varphi$). Let $\mathbb{M} = (W, \mathcal{N}, V)$ be a neighborhood model, w be a world in W, X a set of worlds, and $p \in \Phi$. Then:

Common Core

- $\Vdash_w^{\mathbb{M}} \bot$ iff never
- $\Vdash_w^{\mathbb{M}} p$ iff $w \in V(p)$, for $p \in \Phi$
- $\nVdash_w^{\mathbb{M}} p$ iff $w \notin V(p)$
- $\Vdash_w^{\mathbb{M}} \varphi \vee \psi$ iff $w \in V(\varphi)$ or $w \in V(\psi)$
- $\Vdash_w^{\mathbb{M}} \Delta\varphi$ iff $\Vdash_w^{\mathbb{M}} \neg\nabla\neg\varphi$

Minimal Models, 'e':

- $\Vdash_w^{\mathbb{M}^e} \nabla\varphi$ iff $\{w^* | \Vdash_{w^*}^{\mathbb{M}^e} \varphi\} \in \mathcal{N}(w)$
- $\Vdash_w^{\mathbb{M}^e} \Delta\varphi$ iff $\{W \setminus \{w^* | \Vdash_{w^*}^{\mathbb{M}^e} \varphi\}\} \notin \mathcal{N}(w)$

Supplemented Models, 'm':

- $\Vdash_w^{\mathbb{M}^m} \nabla\varphi$ iff $(\exists X \in \mathcal{N}(w), \forall w^* \in X) : \Vdash_{w^*}^{\mathbb{M}^m} \varphi$
- $\Vdash_w^{\mathbb{M}^m} \Delta\varphi$ iff $(\forall X \in \mathcal{N}(w), \exists w^* \in X) : \Vdash_{w^*}^{\mathbb{M}^m} \varphi$

Quasi-filters, 'r':

- $\Vdash_w^{\mathbb{M}^r} \nabla\varphi$ iff $\mathcal{N}(w) \neq \emptyset$ and $\{w^* | \Vdash_{w^*}^{\mathbb{M}^r} \varphi\} = \mathcal{N}(w)$
- $\Vdash_w^{\mathbb{M}^r} \Delta\varphi$ iff $\mathcal{N}(w) \neq \emptyset$ and $\{W \setminus \{w^* | \Vdash_{w^*}^{\mathbb{M}^r} \varphi\}\} \neq \mathcal{N}(w)$

Augmented, 'k':

- $\Vdash_w^{\mathbb{M}^k} \nabla\varphi$ iff $\{w^* | \Vdash_{w^*}^{\mathbb{M}^k} \varphi\} = \mathcal{N}(w)$
- $\Vdash_w^{\mathbb{M}^k} \Delta\varphi$ iff $\{W \setminus \{w^* | \Vdash_{w^*}^{\mathbb{M}^k} \varphi\}\} \neq \mathcal{N}(w)$.

The following are classical **modal schemata** and frame properties. (More soon on their relationship.) All instances of (**N**), (**C**), and (**M**) are theorems of any normal modal logic. However, none are theorems of classical modal logic. All instances of (**M**) are theorems of monotone logics, and all instances of (**M**) and (**C**) are theorems of regular logics.

To shorten notation, a neighborhood function $\mathcal{N}$ defines a map $N_m : \mathcal{P}(W) \mapsto \mathcal{P}(W)$ such that $N_m(X) = \{w \in W : X \in N(w)\}$, so that $N_m(V(\varphi) = V(\nabla\varphi)$.

(**N**) $\nabla\top$
(n) $\forall w \in W : W \in \mathcal{N}(w)$
(**P**) $\neg\nabla\bot$
(p) $\forall w \in W : \emptyset \notin \mathcal{N}(w)$
(**C**) $\nabla\phi \wedge \nabla\psi \rightarrow \nabla(\phi \wedge \psi)$
(c) $\forall w \in W, \forall X_1, X_2 \subseteq W :$
$(X_1 \in \mathcal{N}(w) \;\&\; X_2 \in \mathcal{N}(w)) \rightarrow X_1 \cap X_2 \in \mathcal{N}(w)$.
(**M**) $\nabla(\phi \wedge \psi) \rightarrow \nabla\phi \wedge \nabla\psi$
(m) $\forall w \in W, \forall X_1, X_2 \subseteq W :$
$(X_1 \subseteq X_2 \;\&\; X_1 \in \mathcal{N}(w)) \rightarrow X_2 \in \mathcal{N}(w)$.
(**D**) $\nabla\phi \rightarrow \Delta\phi$
(d) $\forall x \in W, \forall X \subseteq W : X \in \mathcal{N}(w) \rightarrow -X \notin \mathcal{N}(w)$.
(**T**) $\nabla\phi \rightarrow \phi$
(t) $\forall w \in W, \forall X \subseteq W : X \in \mathcal{N}(w) \rightarrow w \in X$.
(**B**) $\phi \rightarrow \nabla\Delta\phi$
(b) $\forall x \in W, \forall X \subseteq W :$
$w \in X \rightarrow \{W \setminus \{N_m(W \setminus X)\}\} \in \mathcal{N}(w)$
(**4**) $\nabla\nabla\phi \rightarrow \nabla\phi$
(iv) $\forall w \in W, \forall X, Y \subseteq W :$
(**4′**) $\nabla\phi \rightarrow \nabla\nabla\phi$
(iv') $\forall X, Y \subseteq W : X \in \mathcal{N}(w) \rightarrow N_m(X) \in \mathcal{N}(w)$.
$(X \in \mathcal{N}(w) \;\&\; \forall x \in X : Y \in \mathcal{N}(w)) \rightarrow Y \in \mathcal{N}(w)$.
(**5**) $\Delta\phi \rightarrow \nabla\Delta\phi$
(v) $\forall x \in W, \forall X \subseteq W :$
$X \notin \mathcal{N}(w) \rightarrow \{W \setminus N_m(X)\} \in \mathcal{N}(w)$.

Define (**E**) as $\Delta\varphi \leftrightarrow \neg\nabla\neg\varphi$ and consider the following inference rules.

$(RE)\ \frac{\varphi \leftrightarrow \psi}{\nabla\varphi \leftrightarrow \nabla\psi} \quad (RM)\ \frac{\varphi \rightarrow \psi}{\nabla\varphi \rightarrow \nabla\psi} \quad (RR)\ \frac{(\varphi_1 \wedge \varphi_2) \rightarrow \psi}{(\nabla\varphi_1 \wedge \nabla\varphi_2) \rightarrow \nabla\psi}$

Classical modal systems contain (**E**) and are closed under (RE). **Monotone** modal systems are classical but contain all instances of (**M**); equivalently, they contain (**E**) and are closed under (RM). **Regular** modal systems are monotone but contain all instances of (**C**); equivalently, they contain (**E**) and are closed under (RR).

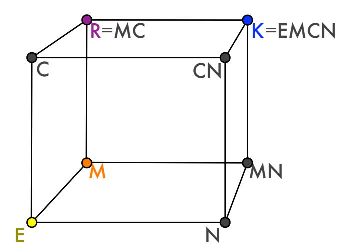

Figure 1: Basic systems of classical modal logic

In addition to the smallest classical modal (E), monotone (EM), regular (EMC), and normal (EMCN) systems, there are eight classical logics defined by combinations of the schemata **M**, **C**, and **N**—each axiomatizable, determined by a class of finite neighborhood models, sound and strongly complete, and decidable (Chellas 1980).

Correspondence Languages

Recall the goal: to define AGM belief revision operators for systems of classical modal logic. The first step of our strategy involves translating classical modal formulas and the relevant neighborhood semantic structure into first-order logic. This section addresses the translation step by appealing to results from modal simulation theory, which identifies a class of neighborhood frames with some multi-modal Kripke frame, and correspondence theory, which here will characterize a bi-modal Kripke frame by sentences of first-order logic. This requires specifying three languages: $\mathcal{L}_\nabla$, a classical propositional monomodal language; $\mathcal{L}_\Diamond$, a standard propositional polymodal language; and $\mathcal{L}^1_\nabla$, the final first-order translation language corresponding to $\mathcal{L}_\nabla$. This technique does not cover all classical modal systems, but it does cover many of them.

Let $p \in \Phi$ and pt be a unary modal operator. A **classical monomodal** grammar and a **standard polymodal** grammar are generated by the following, respectively:

- $\mathcal{L}_\nabla(\Phi) : p \mid \neg\varphi \mid \varphi \vee \psi \mid \nabla\varphi \mid$
- $\mathcal{L}_\Diamond(\Phi) : p \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond_1\varphi \mid \Diamond_2\varphi \mid \mathsf{pt}.$

For the standard polymodal language $\mathcal{L}_\Diamond(\Phi)$, a **a first-order correspondence language** $\mathcal{L}^1_\nabla(\Phi)$ is generated from first-order variables $x, y, z, \ldots$, unary predicates $P_0, P_1, \ldots$ for each propositional atom $p_0, p_1, \ldots \in \Phi$, binary relation symbol(s) R_1, R_2, and a unary relation symbol Q. The set of propositional atoms is constant, so we omit reference to Φ in the remainder.

The Common Core

First-order correspondence languages vary by the conditions imposed on the binary relations, and those conditions are determined by the interpretation supplied to $\Diamond_1$ and $\Diamond_2$ in $\mathcal{L}_\Diamond$ by a standard bi-modal Kripke frame. Otherwise, the translation operations are homomorphic for non-modal formulas.

To focus on this common core, let $\mathcal{F}^2$ be a bi-modal Kripke frame to be defined later for different classes of neighborhood models, and $\mathcal{M} = (\mathcal{F}^2, V)$ a Kripke model based on $\mathcal{F}^2$. Define a translation τ between $\mathcal{L}_\nabla$ and $\mathcal{L}_\Diamond$ on the (non-modal) common core as:

$$\begin{aligned} \bot^\tau &= \bot, \\ p^\tau &= p, \text{for } p \in \Phi, \\ (\neg\varphi)^\tau &= \neg(\varphi^\tau), \\ (\varphi \vee \psi)^\tau &= (\varphi^\tau) \text{ or } (\varphi^\tau). \end{aligned}$$

For every class of classical models, Δ is dual of ∇. So, in principle (E) is part of the common core. But the truth conditions for ∇ can differ by class of models, so the duality principle will depend on the modal simulation for each class of classical models.

Turning to the **local translation** t between $\mathcal{L}_\Diamond$ and $\mathcal{L}^1_\nabla$ for the common core, the unary predicates $P_i \in \mathcal{L}^1_\nabla$ are interpreted by their corresponding propositional variables $p_i \in \Phi$ as follows. $\Vdash^{\mathcal{M}}_w p^t = P(w)$ express that p is satisfied at world w in model $\mathbb{M}$, and this assertion is translated into first-order logic by $P(w)$. $p^t(w)$ abbreviates $\Vdash^{\mathcal{M}}_w p^t = P(w)$; $\neg(p^t(w))$ abbreviates $\nVdash^{\mathcal{M}}_w p^t$. Then:

$$\begin{aligned} (\bot)^t(w) &= x \neq x, \\ (p)^t(w) &= P(w), \\ (\neg\varphi)^t(w) &= \neg(\varphi^t(w)), \\ (\varphi \vee \psi)^t(w) &= \varphi^t(w) \vee \psi^t(w). \end{aligned}$$

Discussion: Intuitively, $(\varphi)^t$ translates the assertion that φ is satisfied at a world within a model. To translate that φ is valid with respect to a class of model, a **global translation** function translates the assertion that φ is satisfied at all worlds with respect to that class of models.

Supplemented models

Supplemented models have been studied extensively, and much is now known about simulation and correspondence for monotone modal logics (Hansen 2003).

Expand the translation function τ between $\mathcal{L}_\nabla$ and $\mathcal{L}_\Diamond$ to cover modal formulas in supplemented models is achieved by defining a bi-frame for the language $\mathcal{L}_{\Diamond_2}$, following (Gasquet and Herzig 1996), where here the modal indices are replaced by mnemonic symbols. Define a bi-modal Kripke frame $\mathcal{F}^2 = (W \cup \wp(W), R_\mathcal{N}, R_\ni, \mathsf{pt})$. The neighborhood function $\mathcal{N} \in \mathbb{F}$ is represented within $\mathcal{F}^2$ by:

$$\begin{aligned} R_\mathcal{N} &= \{(w, X) \in W \times \wp(W) | X \in \mathcal{N}(w)\} \\ R_\ni &= \{(X, w) \in \wp(W) \times W | w \in X\} \\ \mathsf{pt} &= W. \end{aligned}$$

Then, adding equation (1) to the τ-common core $\mathcal{L}_\Diamond$ (i.e., $\mathcal{L}_{\Diamond^m}$) yields a truth preserving translation between the class of supplemented models $\mathbb{M}^m$ and standard bi-modal modal Kripke models based on the class of frames $\mathcal{F}^2$ (Gasquet and Herzig 1996; Kracht and Wolter 1999).

$$(\nabla\varphi)^\tau = \Diamond_\mathcal{N}\Box_\ni(\varphi)^\tau \tag{1}$$

A frame-validity preserving translation $\blacklozenge$ between $\mathcal{L}_{\nabla^e}$ and $\mathcal{L}_\Diamond$ is defined by $\varphi^\blacklozenge = \mathsf{pt} \rightarrow \varphi^t$.

Discussion: From the satisfiability conditions for ∇ and Δ, we can view the monotonic neighborhood function $\mathcal{N}$ to be comprised of two different types of relationships, each represented by a diamond modality. The first, $\Diamond_\mathcal{N}$, expresses when a set of worlds is within the neighborhood associated with a world w, and the second, $\Diamond_\ni$, expresses when w is within a set of worlds. Finally, since the accessibility relations for these two modalities range over worlds and sets of worlds, i.e., $W \cup \wp(W)$, the 0-arity modal constant pt is used to denote the worlds $w \in W$.

Turning to the translation function t between $\mathcal{L}_\Diamond$ and $\mathcal{L}^1_\nabla$, adding equation (2) to the t-common core $\mathcal{L}^1_\nabla$ (i.e., $\mathcal{L}^1_{\nabla^m}$) yields a local truth preserving translation between standard bi-modal Kripke models simulating the class of supplemented neighborhood models and the first-order correspondence language $\mathcal{L}^1_{\nabla^m}$.

$$(\nabla\varphi)^t(w) = \exists x(R_\mathcal{N}wx \wedge \forall y[R_\ni xy \rightarrow \varphi^t(y)]), \tag{2}$$

where R_iab abbreviates $(a, b) \in R_i$.

Finally, the global translation T between $\mathcal{L}_{\nabla^e}$ and $\mathcal{L}_{\Diamond}$ defined by $(\varphi)^T(w) = \forall w(Q(w) \rightarrow (\varphi)^t(w))$ preserves frame validity.

Discussion: By quantifying over subsets of worlds, frame validity expresses a second-order property which does not always admit expression by a first-order formula. In (Kracht 1993; Kracht and Wolter 1999) it was observed that a particular class of classical modal formulas in language $\mathcal{L}_{\nabla}$, interpreted over bi-modal Kripke structures, correspond to Sahlqvist formulas, for which Salvquist correspondence holds via the Sahlqvist-van Benthem algorithm. This technique was extended to monotonic modal logic by Marc Pauly in an unpublished manuscript, which is described in (Hansen 2003). As will be seen in the discussion of minimal models, this result can be applied to some but not all classical modal systems.

Minimal models

Minimal models are the most general class of neighborhood models; this class determines system (E).

To expand the translation function τ between $\mathcal{L}_{\nabla}$ and $\mathcal{L}_{\Diamond}$ with respect to $\mathcal{F}^2$, adding equation (3) to the τ-common core yields a truth preserving translation between the class of minimal models $\mathbb{M}^e$ and standard bi-modal modal Kripke models based on the class of frames $\mathcal{F}^2$ (Gasquet and Herzig 1996).

$$(\nabla\varphi)^{\tau} = \Diamond_{\mathcal{N}}(\Box_{\ni}(\varphi)^{\tau} \wedge \Box_{\mathcal{N}}(\varphi)^{\tau}) \qquad (3)$$

Turning to the translation function t between $\mathcal{L}_{\Diamond^e}$ and $\mathcal{L}^1_{\nabla^e}$, results are limited. This is because the language $\mathcal{L}_{\nabla}$ local translation, equation 4, is not a Sahlqvist formula.

$$(\nabla\varphi)^t(w) = \exists x(R_{\mathcal{N}}wx \wedge [\forall y(R_{\ni}xy \leftrightarrow \varphi^t)]) \qquad (4)$$

Since supplemented models are just the class of minimal models in which all instances of (**M**) are valid, the correspondence results from monotonic modal logic apply. But it is an open question precisely what the classical modal fragment is beyond Pauly's identification of the monotonic modal fragment as the monotonic bisimulation invariant fragment mentioned above; the monotonicity condition of supplemented models is critical in the construction. Recent work has focused on developing an alternative correspondence theory based on a topological semantics (ten Cate, Gabelaia, and Sustretov 2009).

Quasi-filters & augmented models

The list of modal schemata in the previous section are divided into two families, each sound and strongly complete with respect to their associated frames. Let M+S be a propositional monotonic modal system, S a modal schema, then:

1. If $\mathsf{S} \subseteq \{\mathbf{N}, \mathbf{C}, \mathbf{T}, \mathbf{4'}, \mathbf{B}, \mathbf{D}\}$, then MS is sound and strongly complete with respect to the class of monotonic $\mathcal{L}_{\Diamond}$ bi-frames defined by all formulas in S.
2. If $\mathsf{S} \subseteq \{\mathbf{P}, \mathbf{4}, \mathbf{5}\}$, then MS is sound and strongly complete with respect to the class of monotonic $\mathcal{L}_{\Diamond}$ bi-frames defined by all formulas in S.

Briefly, the first family and second family are each sound and strongly complete, but a logic MS such that $\mathsf{S} \subseteq \{\mathbf{N}, \mathbf{C}, \mathbf{T}, \mathbf{4'}, \mathbf{B}, \mathbf{D}\} \cup \{\mathbf{P}, \mathbf{4}, \mathbf{5}\}$ is not necessarily strongly complete (Hansen 2003).

Finally, the class of monotonic $\mathcal{L}_{\nabla}$ bi-frames satisfying condition (c) is defined by the class of supplemented models in which all instances of (**C**) are valid, and the class of $\mathcal{L}_{\nabla}$ bi-frames satisfying both (c) and (n) is defined by the class of supplemented models in which all instances of (**C**) and (**N**) are valid. The former are the class of quasi-filters; the latter the class of augmented models.

AGM

Our goal is to bring AGM belief revision to classical modal logic. We observed in the last section how to construct a first-order correspondence theory for monotonic modal logic using results from modal simulation theory and standard correspondence theory. Now we address the last step, defining AGM revision on this family of correspondence languages, adapting a strategy for normal monomodal logic (Gabbay, Rodrigues, and Russo 2008) which requires (i) a sound and complete axiomatization of the semantics of a classical modal logic, (ii) a classical AGM revision operator.

Recall the AGM postulates (Alchourrón, Gärdenfors, and Makinson 1985) for the revision operator, $*$, where $K = Cn(K)$, and φ, ψ are formulas:

(K$*$1) $K * \phi$ is a belief set.

(K$*$2) $\phi \in (K * \phi)$.

(K$*$3) $(K * \phi) \subseteq Cn(K \cup \{\phi\})$.

(K$*$4) If $\neg\phi \notin K$, then $Cn(K \cup \{\phi\}) \subseteq (K * \phi)$.

(K$*$5) $(K * \phi) = \mathcal{L}^{PL}$ only if $\phi \equiv \bot$.

(K$*$6) If $\phi \equiv \psi$, then $(K * \phi) \equiv (K * \psi)$.

(K$*$7) $K * (\phi \wedge \psi) \subseteq Cn((K * \phi) \cup \{\psi\})$.

(K$*$8) If $\neg\psi \notin (K * \phi)$, then $Cn((K * \phi) \cup \{\psi\}) \subseteq K * (\phi \wedge \psi)$.

Since consistency in classical modal logics is nonclassical, alternatives to (K$*$3) and (K$*$4), and (K*7) and (K*8) are required.

(K$*_{3,4}$) If $K \cup \{p\}$ is consistent, then $K * p = Cn(K \cup \{p\})$;

(K$*_{7,8}$) $Cn((K * p) \cup \{q\}) = K * (p \wedge q)$, when q is consistent with $K * p$.

Because classical modal logics are extensions of classical logic, our correspondence language essentially maps the satisfiability conditions of modal formulas into corresponding first-order predicates along with additional first-order formulas that express the corresponding neighborhood frame conditions. Thus, the consistency condition is still classical consistency, but the arguments will be the first-order translations along with any additional formulas needed to characterize the frame properties of a modal system.

Turn to the **definition of AGM revision** in EM. Let $\Lambda^t(w)$ be the first-order local translation into $\mathcal{L}^1_{\nabla}$ of a classical monotonic modal theory, $\phi^t(w), \psi^t(w)$ first-order local translations of classical monotonic modal formula, and

$\mathfrak{N}_{\mathsf{M.S}}$ the (possibly empty) first-order characterization of classical monotonic modal system M.S. Then:

$$\Lambda *_{\mathsf{m}} \psi = \{\phi : \Lambda^t(w) *_a \psi^t(w) \wedge \mathfrak{N}_{\mathsf{M.S}} \vdash \phi^t(w)\}.$$

We now have the following result. Proof is in the appendix.

Theorem 01 *The operator $*_{\mathsf{m}}$ is an AGM operator.*

There are two families of revision operators for monotone modal logic, which we may generalize.

Corollary 02 *For any system of monotonic modal logic* EM.S*:*

1. *if $\mathsf{S} \subseteq \{\mathbf{N}, \mathbf{C}, \mathbf{T}, \mathbf{4'}, \mathbf{B}, \mathbf{D}\}$, there is an operator $*_{\mathsf{m.s}}$ that is AGM.*
2. *if $\mathsf{S} \subseteq \{\mathbf{P}, \mathbf{4}, \mathbf{5}\}$, there is an operator $*_{\mathsf{m.s}}$ that is AGM.*

An Application

One application is to interpret the necessity operator ∇ as 'qualitative judgment of high likelihood' in system (EMN) (Arló-Costa 2005) in general, or as qualitative judgments of high evidential probability in particular (Kyburg and Teng 2002). In the case of evidential probability (EP), probability is assigned to a sentence based upon both logical and probabilistic information—and there is a small chance that each item of evidence is accepted in error. But, some evidence sets containing an error are better than others, which is to say that some evidence for a statement is more robust than other evidence. To assess robustness of an EP assignment to a statement, one needs to look at a set of counter-factual EP probability assignments (Haenni et al. 2009) to measure the variation in probability assignments when various items of evidence are excluded because false. A counterfactual evidence set (relative to a statement) is determined by the contraction operator $\dot{-}_{\mathsf{mn}}$ defined by the Harper Identity (Harper 1977) with respect to $*_{\mathsf{mn}}$.

Caution should be exercised when the modality ∇ is interpreted as a generic epistemic operator. See in particular the negative results of (Hansson 1999) concerning epistemic modal revision.

Complexity / Limits to the Approach

The satisfiability problem for classical systems without schema (C) is in NP, PSPACE otherwise; for multi-modal normal systems it is PSPACE-complete. Hence, the method described here is intended primarily to be exploratory.

The main bottleneck in this approach is the failure to have a complete first-order correspondent for classical modal logic. But, even if we did know the first-order classical modal fragment, it is likely that this fragment would not cover all of classical modal logic. More is known about simulation. So, a natural line of work would be to define the AGM operations directly for multi-modal normal logics, but even this would not yield a complete AGM theory for classical modal systems.

This said, the technique should not be under-appreciated. Aside from AGM, the general technique holds promise for importing a variety of revision operators into monotone modal systems, which promises to open the study of belief change in a variety of 'non-adjunctive' systems.

Appendix

Proof of Theorem 01 The operator $*_{\mathsf{m}}$ is an AGM operator for the smallest monotonic logic, EM

Let Λ be an EM-consistent monotonic modal theory, and ϕ, ψ and γ sentences in $\mathcal{L}_\nabla$. We show that $*_{\mathsf{m}}$ satisfies the AGM postulates. First, observe that M is the smallest classical monotonic modal system, which is equivalent to EM.S, where $\mathsf{S} = \emptyset$. Hence, $\mathfrak{N}_{\mathsf{M.S}} = \emptyset$.

1. $(\mathbf{\Lambda}*\mathbf{1})$: $\Lambda *_{\mathsf{m}} \phi$ is a belief set.

   Since $\Lambda^t(w) * (\phi^t(w) \wedge \mathcal{A}_{\mathcal{L}_\nabla}))$ is closed under $\vdash$ by (K$*$1), then $\Lambda *_{\mathsf{m}} \phi$ is closed under $\vdash_{\mathsf{EM}}$.

2. $(\mathbf{\Lambda}*\mathbf{2})$: $\phi \in (\Lambda *_{\mathsf{m}} \phi)$.

   From $(K*2)$ we have $\phi^t(w) \wedge \mathcal{A}_{\mathcal{L}_\nabla} \in \Lambda^t(w) * (\phi^t(w) \wedge \mathcal{A}_L)$. Since $\Lambda^t(w) * (\phi^t(w) \wedge \mathcal{A}_{\mathcal{L}_\nabla})$ is closed under $\vdash$, by (K$*$1), and $\vdash$ is reflexive, then $\Lambda^t(w) * (\phi^t(w) \wedge \mathcal{A}_{\mathcal{L}_\nabla}) \vdash \phi^t(w)$. So, $\phi \in (\Lambda *_{\mathsf{m}} \phi)$ by $(\Lambda*2)$.

3. $(\mathbf{\Lambda}*\mathbf{3}, \mathbf{4})$: If sentence ϕ is EM-consistent with Λ, then $\Lambda *_{\mathsf{m}} \phi$ is equal to the closure of $\{\Lambda \cup \{\phi\}\}$ under $\vdash_{\mathsf{EM}}$, written $C_{\mathsf{m}}(\Lambda \cup \{\phi\})$.

   First we make the following two observations.

   Observation 1. Recall that if Λ is an EM-consistent modal theory, then $\Lambda \nvdash_{\mathsf{EM}} \bot$ and there exists a monotone neighborhood model for Λ.

   Observation 2. If $\Lambda \cup \{\phi\}$ is consistent with respect to classical modal logic EM, then $\Lambda^t(w)$ is classically consistent with respect to its translation, $\phi^t(w) \wedge \mathcal{A}_{\mathcal{L}_\nabla}$. Since by hypothesis $\Lambda \cup \{\phi\}$ has a monotone neighborhood model, by Observation 1, there exists a classical first-order model of its translation, $\Lambda^t(w) \cup \{\phi^t(w) \wedge \mathcal{A}_{\mathcal{L}_\nabla}\}$.

   Suppose that Θ denotes the classical provability closure of the first-order translation from Observation 2, $\Lambda^t(w)*(\phi^t(w) \wedge \mathcal{A}_{\mathcal{L}_\nabla})$. We now show that if $\psi^t(w) \in \Theta$, then $\Lambda *_{\mathsf{m}} \phi \vdash \psi$.

   Suppose that $C_{\mathsf{m}}(\Lambda)$ is Λ closed under $\vdash_{\mathsf{EM}}$ and $\Lambda^t(w)$ is the first-order translation of Λ. We denote the corresponding $\mathcal{A}_{\mathcal{L}_\nabla}$-simulated closure in classical logic of the first-order translation by $Cn(\Lambda^t)$. There are two parts.

   (a) First, for any $\gamma \in \mathcal{A}_{\mathcal{L}_\nabla}$, if $\gamma^t \in Cn(\Lambda^t)$, then $\gamma \in \Lambda$. To see this, notice that $C_{\mathsf{m}}(\Lambda)$ is a maximally EM-consistent set, so $\gamma \in C_{\mathsf{m}}(\Lambda)$ iff $\Lambda \vdash_{\mathsf{EM}} \gamma$.
   Proof: Suppose that $\gamma \notin \Lambda$. Then, there is a classical monotone model satisfying $\Lambda \cup \{\neg\gamma\}$ and a translation of this into first-order logic. But on the first-order model for this translation $\gamma^t \notin Cn(\Lambda^t)$, which falsifies the hypothesis.

   (b) Second, for a closed classical theory $Cn(\Lambda^t)$ s.t. $\mathcal{A}_{\mathcal{L}_\nabla} \subseteq Cn(\Lambda^t)$ and $\{\gamma : \gamma^t \in \Lambda^t\}$, then $\Lambda \vdash \gamma$ only if $\gamma^t \in Cn(\Lambda^t)$.
   Proof: Suppose that $\gamma^t \notin Cn(\Lambda^t)$. Then there is a model of $\Lambda^t \cup \{\neg\gamma^t\}$, so there is classical monotone model satisfying $\Lambda \cup \{\neg\gamma\}$ which falsifies the hypothesis.

This concludes the proof of $(\mathbf{\Lambda}*\mathbf{3}, \mathbf{4})$.

4. $(\mathbf{\Lambda *5})$: $\Lambda *_{\mathsf{m}} \phi = \mathcal{L}_{\nabla}$ only if $\phi \equiv \bot$.

   Since Λ is an EM-consistent modal theory, $\Lambda \neq \mathcal{L}_{\nabla}$. So $\Lambda^t(w) \neq \mathcal{L}^1_{\nabla}$. So if $\Lambda^t(w) * \phi^t(w) = \mathcal{L}^1_{\nabla}$, then $\phi^t(w) = \bot$; thus $\phi \equiv \bot$.

5. $(\mathbf{\Lambda *6})$: If $\vdash_{\mathsf{EM}} \phi \equiv \psi$, then $\Lambda *_{\mathsf{m}} \phi \equiv \Lambda *_{\mathsf{m}} \psi$.

   If $\vdash_{\mathsf{EM}} \phi \equiv \psi$, then $\vdash \phi^t \wedge \mathcal{A}_{\mathcal{L}_{\nabla}}) \equiv \psi^t \wedge \mathcal{A}_{\mathcal{L}_{\nabla}}$. So, by (K$*$6), $\Lambda * (\phi^t \wedge \mathcal{A}_{\mathcal{L}_{\nabla}}) \equiv \Lambda * (\psi^t \wedge \mathcal{A}_{\mathcal{L}_{\nabla}})$. Therefore, $\Lambda *_{\mathsf{m}} \phi \equiv \Lambda *_{\mathsf{m}} \psi$.

6. $(\mathbf{\Lambda *7, 8})$: $\Lambda *_{\mathsf{m}} (\phi \wedge \psi) = C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\})$, when ψ is EM-consistent with $\Lambda *_{\mathsf{m}} \phi)$.

   Now we proceed in two parts.

   (a) $\Lambda *_{\mathsf{m}} (\phi \wedge \psi) \subseteq C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\})$: By $(\Lambda *1)$, $\Lambda *_{\mathsf{m}} (\phi \wedge \psi) = C_{\mathsf{m}}(\Lambda *_{\mathsf{m}} (\phi \wedge \psi))$. Suppose that $\gamma \in C_{\mathsf{m}}(\Lambda *_{\mathsf{m}} (\phi \wedge \psi))$. Then by the correspondence theorem $\gamma^t \in Cn(\Lambda^t * (\phi^t \wedge \psi^t \wedge \mathcal{A}_{\mathcal{L}_{\nabla}}))$. So $\gamma^t \in Cn(\Lambda^t * (\phi^t \wedge \mathcal{A}_{\mathcal{L}_{\nabla}}) \cup \{\psi^t\})$, by (K$*$7), and $\gamma \in C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\})$, by correspondence. Since γ is an arbitrary modal formula, $\Lambda *_{\mathsf{m}} (\phi \wedge \psi) \subseteq C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\})$.

   (b) $C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\}) \subseteq \Lambda *_{\mathsf{m}} (\phi \wedge \psi)$: Suppose that $\gamma \in C_{\mathsf{m}}(\Lambda *_{\mathsf{m}} \phi)$. Since γ is EM-consistent with $\Lambda *_{\mathsf{m}} \phi)$, $\gamma \in C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\})$. Thus, $\gamma^t \in Cn(\Lambda^t * (\phi^t \wedge \mathcal{A}_{\mathcal{L}_{\nabla}}) \cup \{\psi^t\})$, by the correspondence theorem, and $\gamma^t \in Cn(\Lambda^t * (\phi^t \wedge \psi^t \wedge \mathcal{A}_{\mathcal{L}_{\nabla}}))$, by (K$*$8). So, $\gamma \in C_{\mathsf{m}}(\Lambda *_{\mathsf{m}} (\phi \wedge \psi))$, by correspondence. Since γ is an arbitrary modal formula, $C_{\mathsf{m}}((\Lambda *_{\mathsf{m}} \phi) \cup \{\psi\}) \subseteq \Lambda *_{\mathsf{m}} (\phi \wedge \psi)$.

Proof of Corollary 2 Example proof omitted for space.

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