Applied Logic without Psychologism

Gregory Wheeler

Artificial Intelligence Center - CENTRIA
Department of Computer Science, Universidade Nova de Lisboa
2829-516 Caparica, Portugal
grw@fct.unl.pt

Abstract. Logic is a celebrated representation language because of its formal generality. But there are two senses in which a logic may be considered general, one that concerns a technical ability to discriminate between different types of individuals, and another that concerns constitutive norms for reasoning as such. This essay embraces the former, *permutation-invariance* conception of logic and rejects the latter, Fregean conception of logic. The question of how to apply logic under this pure invariantist view is addressed, and a methodology is given. The pure invariantist view is contrasted with logical pluralism, and a methodology for applied logic is demonstrated in remarks on a variety of issues concerning non-monotonic logic and non-monotonic inference, including Charles Morgan's impossibility results for non-monotonic logic, David Makinson's normative constraints for non-monotonic inference, and Igor Douven and Timothy Williamson's proposed formal constraints on rational acceptance.

1 Introduction

Psychologism in logic refers to a variety of views about the relationship between psychology and logic, but its traditional form holds that the laws of logic are grounded in psychological facts. So, the rules of logic yield the laws of thought by virtue of our psychological composition. This view is attributed to John Stuart Mill, among others, and was assailed by Frege.

According to Frege the plausibility of Millian psychologism trades on an ambiguity in the phrase 'law of thought'. The reading necessary for Mill's view entails that logical laws govern thinking in the same manner that physical laws govern physical events, which means that logic is treated as a part of descriptive psychology. A logical rule of inference then is an abstraction from the psychological activity of drawing a demonstrative inference. But logic for Frege is a normative, substantive science, and no set of laws can be both normative *and* descriptive. So, Millian psychologism is false.

The point to stress is that despite Frege's opposition to Millian psychologism, he nevertheless endorses the view that logical laws reveal how one should think.

> Any law asserting what is, can be conceived as prescribing that one ought to think in conformity with it, and is thus in that sense a law of thought.

> This holds for laws of geometry and physics no less than for laws of logic. The latter have a special title to the name 'laws of thought' only if we mean to assert that they are the most general laws, which prescribe universally the way in which one ought to think if one is to think at all [9, p. xv].

So, by Frege's lights, the problem with Millian psychologism is that it is ill-suited for the purpose of logic, which is to reveal how one ought to think. Viewing logic to provide constitutive norms for thought is the *sine qua non* of Frege's view of logic, which by contemporary standards is a form of psychologism.

Fregean psychologism is the target of this essay. It is to be contrasted with contemporary mathematical conceptions of logic that characterize logical laws and concepts as 'topic neutral'. On the contemporary view logical notions are not identified as those that provide constitutive norms of thought as such, but rather are just those concepts that are invariant under the widest possible group variations [17, 32, 22, 30]. Since the identity of the objects in the domain is irrelevant, logic is no more *about* thought as such than solid geometry is about solid bodies as such. Each provides a vocabulary of concepts and facts about their necessary relations that in turn may be applied to represent a line of argument, or the extension of objects. But successful application of a logic or a geometry to a domain is a question answered (in part) by appealing to facts and considerations entirely outside of the logic, or outside of the geometry as the case may be.

We may crystalize this difference between what I call Fregean psychologism and invariantist views of logic by appealing to John MacFarlane's distinction between notions of logical formality [18, p. 51].

- **1-formality**: Logic is *1-formal* when it provides constitutive norms for thought as such, which is contrasted to norms of thought about a particular subject. 1-formal laws are normative laws to which any conceptual activity—asserting, inferring, supposing, judging—must be held to account.

- **2-formality**: Logic is *2-formal* when its characteristic laws and concepts are indifferent to the particular identities of different objects. 2-formal laws treat each object the same. Mathematically, 2-formality is defined as invariance under all permutations of the domain of objects.

Fregean psychologism follows from a commitment to viewing logic to be, necessarily, 1-formal. Some, including MacFarlane, think that both 1-formality *and* 2-formality are necessary features of logic. The anti-psychologism I am advocating is based upon the view that logic is 2-formal, and that 1-formality is neither necessary nor needed. Call this position *pure invariantism*.[1]

[1] By 'logic' one may mean the study of logical concepts, which are purely invariant under some group transformation or another, or mean the identification of logically true sentence of some system of logic or another. Invariantism is a thesis about logical concepts, not a thesis about logically true sentences.

There is a benefit and a problem accompanying my anti-psychologism. The benefit is that since logic is a part of mathematics, the principles for applying logic are no different than the principles for applying any other branch of mathematics. This includes applications of logic in formal epistemology, economics, and artificial intelligence, each of which discusses rules for governing the 'rational behavior' of agents. One should apply logic in these domains without appealing to psychologism, for only by viewing logic in terms of 2-formality is one in a position to properly evaluate various proposals found in these fields. We will return to this point in section 2.

The problem with pure invariantism is foundational in nature, and may appear to threaten the benefit just outlined. Invariance methods for distinguishing logical concepts from non-logical concepts are widely viewed to be necessary but not sufficient conditions for logicality. One reason is that the classical invariantist criterion is characterized as invariance under bijections, but this only connects up sets of the same size and fails to account for the behavior of logical concepts across sets of different cardinalities. Also, any notion defined in terms of cardinalities is invariant under bijections, but not all so-defined notions are plausibly viewed to be logical.

These observations have led to several notions of invariance, rather than one, which have generated several notions of logicality, each of varying strength. This suggests that logicality comes in (not necessarily comparable) degrees. But since permutation invariance does not yield a unique classification of logicality, some have argued that an additional criterion is needed to yield *a principled* distinction between logical and non-logic concepts. The threat then is that there is an unprincipled conventionalism at the core of the invariantist view [18], which leads one to a logical pluralist view about consequence [2] and a dissolution of the norms for inference.

For example, MacFarlane has argued for a neo-Fregean view of logicality that selects the 'intrinsic structure' of logical types preserved under permutation as just those that capture the capacities of propositional contents to be used in assertion and inference as such [18, p. 224]. But the problem of principled demarcation and this solution is bound up in the very same conception of logic, namely that the normative force of logical notions is binding on conceptual activity as such. Rather than offer a neo-Fregean solution to a general problem facing the invariantist conception of logic, MacFarlane has offered a neo-Fregean solution to the Fregeans' problem with invariantism.

On the pure invariantist view, constitutive norms for logicality are not general norms for thought as such, so there is no general, normative psychological basis for selecting intrinsic logical structure. This point will be addressed in section 3, where I will illustrate the mischief caused by the notion of 1-formality in a discussion of non-monotonic logic. This extended discussion highlights a methodology for applying logic without psychologism.

2 On the Application of Logic

The aim of applied logic is similar in kind to the aim of other branches of applied mathematics, which is to select a formal structure that suitably represents a particular problem domain. Although the paradigmatic domain for logic is mathematics itself, logics and logical methods are also applied in various non-mathematical domains, such as formal epistemology, economics, and computer science. Formal epistemologists want to give precise meaning to epistemic concepts and the relationships that hold among these notions when making, or changing, an epistemic evaluation of a belief. Economists want to model the interaction of rational agents within different decision environments. Computer scientists want to devise systems that will, like people, go beyond what is entailed by the evidence, that will focus on relevant conclusions, that will accommodate many *arguments* that do not conform to the classical deductive model, but that people regard as 'good'.

The method of applying logic is no different than the method of applying any other branch of mathematics. If I observe a set of objects that is invariant under the group of similarity transformations, then I am in a position to apply the axioms of Euclidean geometry to evaluate sentences expressing transformations of the objects in that set. If I find instead that the set is invariant under the group of affine transformations, then I am in a position to apply the more general axioms of affine geometry to that set. Whether it is *better* to use Euclidean geometry or affine geometry to model the transformations that are preserved on a set of objects is not a question for geometry. Rather, settling this question depends upon the features of the set of objects, what I want to *do* with the representation of those objects, and what resources I have at my disposal. A classifier for satellite images may be insensitive to the differences between these two systems of geometry, for example, depending upon whether the classification task needs to distinguish between types of triangles or triangles as such. This is a preference we impose upon geometry, not a preference geometry imposes upon us.

The difference in granularity between Euclidean geometry and affine geometry therefore does not yield prescriptive laws of thought about Euclidean and affine groups of objects as such. Saying that our judgments, suppositions, or inferences about a set of objects may change after understanding that a theorem is applicable to a set of objects is one thing. But saying that a theorem yields constitutive norms for judgments, suppositions, or inferences about that set of objects is quite another matter. Mathematical results alone do not have this normative power.

This point generalizes. If a set of objects in the world is equipped with a binary function that makes it a group, and there is a theorem of group theory that says every object of a group satisfies F, then F holds of this set of objects equipped with that function. What attitude you should take toward those objects with respect to F is not a mathematical question. Theorems of logic are no different. If a structure satisfies the hypothesis of a theorem, then that structure satisfies the conclusion of that theorem too. What attitude you should take toward these facts is not a matter legislated by logic.

On this conception of logic and the methodology for its application there are three factors that enter into evaluating an applied logic. The first two take stock of the structural features of the logic and those of the problem domain, respectively, for you must judge how well the structural features of the logic match the salient features of the problem domain (or concept) that you wish to represent. For example, the monadic modal operator $\Box$, interpreted within a basic normal modal logic, distributes freely over boolean conjunction, since

$$\Box(\phi \land \psi) \leftrightarrow (\Box\phi \land \Box\psi)$$

is a theorem of every normal modal logic. So, the distribution properties of the class of normal frames projects this distribution property onto the space of possible representations of monadic modalities within this class of logics.

But just because normal modal operators freely distribute over conjunction doesn't mean that all modal concepts freely distribute over conjunction. Suppose, for example, that we wanted to represent the modality 'has high probability' by the $\Box$ within our basic modal language. It would *not* be suitable to use the class of normal modal logics for a modal logic of 'has high probability' however, since

$$(\Box\phi \land \Box\psi) \rightarrow \Box(\phi \land \psi)$$

is false for 'has high probability': the joint measure of $(\phi \land \psi)$ may be strictly less than the high measure for ϕ and the high measure for ψ.

The problem is that Fregean psychologism instills the wrong habit of mind, since imagining that logical properties provide constitutive norms for conceptual activity as such precludes assessing the fit between the logic and a problem domain. If you view the K axiom to be a constitutive norm for modality, then the failure of 'has high probability' to distribute freely across conjunction entails more than simply that this predicate isn't properly classified as a modality. It also follows that your thoughts of modalities as such should render 'has high probability' incoherent. It would not make sense on this view to investigate a minimal-model semantics [5, Ch. 7], for example, in which this distribution property fails to hold, nor to investigate whether the resulting modal logic provided an appropriate representation for the modality 'has high probability'.

Reconciling the constraints of a problem domain with those of a formal framework is difficult precisely because there are no first principles that function in the manner imagined by proponents of 1-formality. There are no normative logical laws of thought. Instead, we must compare the structural constraints of the problem domain to the logical structure of the representation language before investing normative weight to the logical truths of that language. And his point carries through to investigations of logicality itself and its relationship to normativity. For instance, Ken Warmbrod [33] proposes a two-tiered approach to defining logicality, a minimal core logical theory that remains fixed, and several extensions that are devised for practical reasons. But the logical concepts within this conservative core can be further ordered in terms of their logical content. For example, on one account the universal quantifier and the conditional are the primary logical notions [3], whereas on another existential quantification and disjunction are primary [4], and thus 'more logical' than the former pair. These studies of logical notions reveal facts about logicality as such, and one of them is that there isn't a single core to be found. They do not reveal facts about normativity.

Pure invariantism is also distinct from logical pluralism, which holds that logical consequence is the principle concern of logic and that there is more than one consequence relation, all on par [2]. There are several ways to understand pluralism, but at least one way of doing so is uncontroversial. Consider: A formula ψ is a logical consequence of ϕ just when every distribution of truth values that satisfies ϕ satisfies ψ. Now, we may consider whether ψ is a syntactic consequence of ϕ when we have resources of a proof theory that effects derivation of ψ in ϕ, or consider ψ a semantic consequence of ϕ when all models of ϕ are models of ψ. We might also consider whether ϕ is a consequence of ψ by referring to a set of finitely axiomatized sentence schemata along with a set of validity preserving inference rules. Notice, however, that considered as a branch of mathematics, there is nothing that requires that every system of these types characterize one and only one consequence relation. Indeed, there are several systems.

And so we have a class of non-equivalent logical consequence relations that are associated with different systems of logic. Some of these systems of logic are more mature than others, by which I mean that some logics have better developed model theories and proof theories than others do. Intuitionistic, relevance and linear logics are interestingly different types of constructive propositional logics, and each of them is different from classical logic. These facts about various systems of logic and their consequence operations are all on par in the sense that they are known results. If this were all that was meant by logical pluralism, then everyone would be (and should be) a pluralist.

But on Restall and Beall's conception of logical consequence, preservation of truth occurs within a model but there is more than one way to specify a model—each yielding an 'equally good' consequence relation. This means that for an argument valid on one logic in this class but invalid on another 'there is no further fact of the matter as to whether the argument is *really* valid' [29]. Thus, this view of pluralism embraces a substantive thesis about the structure of arguments as such. But Restall and Beall's pluralism is too permissive about validity, since there may well be good *logical* reasons to favor one conception of logical consequence over others if one includes evidence from branches of logic other than model theory. The problem with logical pluralism is that it embraces a too-permissive view about valid arguments which is licensed by Restall and Beall's restricted view of logic as being principally about logical consequence. So, in effect Beall and Restall ask us to embrace a radical thesis about the under-constrained nature of argument validity as well as ignore evidence from other branches of logic that provide indirect insights to questions about argument validity. Pure invariantism does not follow this advice.

This brings us to the third and last factor necessary for evaluating applied logics. The expressive capabilities of a formal language are typically inversely related to the inferential properties of the proof system for that language. The more you can say, the less you can do. Non-logical problem domains—such as natural language—compound this problem because they very often are much richer than even the most expressive formal languages [16, 27]. Hence, in nearly all applications of logic, one must select what features of the problem domain to represent within the framework and what features to ignore.

The point to stress is that reconciling the structure of a logic to the structure of a problem domain is mediated by what one wants to do with the formal representation. We want a mathematical account of why logical notions are insensitive across various sized sets, for instance, so we investigate notions of permutation-invariance that aim to preserve *those* properties. A formal language that a semanticist might find insightful may be useless to a computational linguist. Likewise, restricting oneself to the class of decidable logics would often be of little interest to a philosopher interested in a formal analysis of some part of natural language, such as generics or conditionals.

The benefit of the invariantist picture of logic and this method of application is that the evaluation criteria are no different from any other use of mathematics. This point is particularly relevant to applications of logic in formal epistemology and computer science. While there are several features of belief fixation and belief change that should give one pause to the claim that logical methods are the right choice for modeling rational belief kinematics, the act of proposing a logic for such a task does not commit one to conceptual confusion: we understand both how to formulate the problem and also how to judge the merits of candidate solutions. The methodology of applying logic simply parameterizes the task. So there is no category mistake involved in applying logic to model human reasoning, as Gilbert Harman has charged [12, 13]. Instead it is a methodological error to think that logic itself yields constitutive norms for human inference. Harman's 'inference-implication' fallacy applies to applications of logic that endorse Fregean psychologism, not to applied logic as such. While it is a mistake to regard the study of logics as the study of norms for human inference, it is likewise mistaken to regard applied logics within formal epistemology to be predicated on this methodological error.

3 Logical Structure and Non-monotonic logics

Because logic does not provide constitutive norms for cognitive activity as such, logicality is simply a parameter describing the permutation-invariability of certain notions of a formal language. These features of a language should be decided upon before applying the logic, but the reasons for choosing one sense of logicality over another depends upon what one wishes to do with the language. Logicality is a technical parameter, not a talisman.

Fregean psychologism instills a mistaken habit of mind when thinking about logic and its use. This point may be illustrated by considering the relationship between *defeasible inference* and its formal counterpart, *non-monotonic consequence relations*. I want to focus on a recent impossibility result directed against non-monotonic logics to illustrate the mischief Fregean psychologism creates by licensing attractive formal properties of a logical system as normative constraints on cognitive activity. The aim of this section is to demonstrate how normative questions of *logicality* and *what structural properties are correctly ascribed of a subject* are not answered simply by appealing to the properties of a candidate logic, motivated by abstract principles of rationality, but are instead addressed by following the methodology outlined in section 2.

3.1 Belief structures and evidential relations

The view that there are non-monotonic argument structures goes at least as far back as R. A. Fisher's [8, 7] observation that statistical reduction should be viewed as a type of *logical*, non-demonstrative inference.

Unlike demonstrative inference from true premises, the validity of a non-demonstrative, uncertain inference can be undermined by additional premises: a conclusion may be drawn from premises supported by the total evidence available now, but new premises may be added that remove any and all support for drawing that conclusion.

This behavior of non-demonstrative inference is neither shared nor desired in mathematical arguments. If a statement is a logical consequence of a set of premises, that statement remains a consequence however we might choose to augment the premises. Once a theorem, always a theorem, which is why a theorem may be used as a lemma: even if the result of a lemma is misused in an incorrect proof, the result of the lemma remains. We do not have to start the argument all over again from scratch.

How then can a logic fail to be monotone?

Before taking up this question, recall that logical structure can mean many things. In one important sense it refers to the uniform substitution of arbitrary boolean formulas for elementary formulas within any formula in the language. This sense of logical structure marks an important limit point. But to express this, we first need to define supraclassical consequence. Suppose $\mathcal{L}$ is a language and $\Phi_{\mathcal{L}}$ is the set of formulas of $\mathcal{L}$. Then a consequence relation in $\mathcal{L}$ is a subset, $C \subseteq \mathcal{P}(\Phi_{\mathcal{L}}) \times \Phi_{\mathcal{L}}$. Let Cn be classical consequence: $(\Gamma, \varphi) \in Cn$ if and only if $\Gamma \vdash \varphi$. Then $C \subseteq \mathcal{P}(\Phi_{\mathcal{L}}) \times \Phi_{\mathcal{L}}$ is supraclassical if and only if $Cn \subseteq C$.

The reason that this notion of logical structure is important is that logical consequence is maximal with respect to uniform substitution: there is no nontrivial supraclassical closure relation on a language $\mathcal{L}$ that expresses logical consequence that is closed under uniform substitution except for logical consequence [20, p. 15].

Returning to our question about whether all logics are monotonic, some authors appear to have this notion of logical structure in mind when considering the plausibility of a non-monotonic logic. For example, Charles Morgan [23, 24] has argued that non-monotonic logic is impossible: a consequence relation must be weakly, positively monotone [24, p. 326].[2]

But rather than frame his notion directly in terms of a substitution function over Boolean languages, Morgan aims for a more general result independent of the structure of a language. He attempts this by framing his main argument against non-monotonic consequence in terms of *belief structures*. A belief structure is a semi-ordered set whose elements are sets of sentences of a language $\mathcal{L}$. The focus of interest is the ordering relation *LE* defined over arbitrary sets, Γ and Δ. The idea is that a set of sentences is intended to represent 'an instantaneous snapshot of the belief system of a rational agent' while Γ *LE* Δ expresses that 'the degree of joint acceptability of the members [sentences] of Γ is less than or equal to the degree of joint acceptability of the members [sentences] of Δ' [24, p. 328]. On this view, a logic **L** is a set of arbitrary *rational* belief structures $\{LE_1, LE_2, ...\}$, where a rational belief structure is a subset of $\mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ satisfying

> **Reflexivity**: Γ *LE* Γ,
> **Transitivity**: If Γ *LE* Γ' and Γ' *LE* Γ'', then Γ *LE* Γ'', and
> **The Subset Principle**: If $\Gamma \subseteq \Delta$, then Δ *LE* Γ.

Finally, soundness and completeness properties for **L** with respect to a provability relation $\vdash_b$ are defined as follows:

> **Soundness**: If $\Gamma \vdash_b A$, then Γ *LE* $\{A\}$ for all (most) rational belief structures $LE \in \mathbf{L}$.
>
> **Completeness**: If Γ *LE* $\{A\}$, then $\Gamma \vdash_b A$ for all (most) rational belief structures $LE \in \mathbf{L}$.[3]

Morgan's proposal, then, is for **L** to impose minimal prescriptive restrictions on belief structures to pick out those that are rational:

> Each distinct logic will pick out a different set of belief structures; those for classical logic will be different from those for intuitionism, and both will be different than those for Post logic. But the important point is that from the standpoint of Logic **L**, all and only the belief structures in **L** are rational [24, p. 329].

Hence, if every arbitrary set of rational belief structures is monotonic, then every logic must be monotonic as well—which is precisely the result of Morgan's Theorem 1.

[2] In [24] there are actually 4 theorems given which aim to establish this impossibility result, viewed as an argument by four cases. The result we discuss here is the first of these theorems, and is first offered in [23]. It is the most general of his arguments.

[3] The inclusion of 'most' in each construction appears only in (2000). Morgan states that a corollary to Theorem 1 may be established if 'most' means more than 50% [24, p. 330].

Theorem 1. *Let $\boldsymbol{L}$ be an arbitrary set of rational belief structures which are reflexive, transitive, and satisfy the subset principle. Further suppose that logical entailment $\vdash_b$ is sound and complete with respect to the set $\boldsymbol{L}$. Then logical entailment is monotonic; that is, if $\Gamma \vdash_b A$, then $\Gamma \cup \Delta \vdash_b A$.*

Proof. (Morgan 2000):

1. $\Gamma \vdash_b A$	given.
2. $\Gamma\ LE\ \{A\}$ for all $LE \in L$.	1, soundness.
3. $\Gamma \cup \Delta\ LE\ \Gamma$ for all $LE \in L$.	subset principle
4. $\Gamma \cup \Delta\ LE\ \{A\}$ for all $LE \in L$.	2, 3 transitivity.
5. $\Gamma \cup \Delta \vdash_b A$	4, completeness.

The normative force of Theorem 1 rests upon the claim that *reflexivity, transitivity* and *the subset principle* are appropriate rationality constraints for belief structures.

Consider the case for the subset principle. Morgan argues that

> [the subset principle] is motivated by simple relative frequency considerations....[A] theory which claims both A *and* B will be more difficult to support than a theory which claims just A. Looking at it from another point of view, there will be fewer (or no more) universe designs compatible with both A and B than there are compatible with just A. In general, if Γ makes no more claims about the universe than Δ, then Γ is at least as likely as Δ [24, p. 329].

Morgan's argument appears to be that a set of sentences Δ has fewer compatible universe designs than any of its proper subsets, so Δ is less likely to hold than any of its proper subsets. Hence, Δ is harder to support than any of its subsets. Therefore, the degree of support for a set Δ is negatively correlated with the number of possible universe designs compatible with Δ.

There are, however, two points counting against this argument. First, it is misleading to suggest that the subset principle is motivated by relative frequency considerations. Morgan is not referring to repetitive events (such as outcomes of a gaming device) but rather to the likelihood of universe designs, for which there are no relative frequencies for an epistemic agent to consider. Second, even though there are no more universe designs that satisfy a given set of sentences than with any of its proper subsets, this semantic feature of models bears no relationship to the degree of joint acceptability that may hold among members of an arbitrary set of sentences: 'Harder to satisfy' does not entail 'harder to support', and Morgan's equivocation between these two senses of 'acceptability' is fatal to his argument. Relations such as *prediction* and *justification* are classic examples of epistemic relations between sentences that bear directly on the joint epistemic acceptability of a collection of sentences but do not satisfy the subset principle.

To see this last point, recall the notion of joint acceptability that underpins the interpretation of LE. The subset principle is therefore equivalent to the following proposition:

Proposition 1. *If a set of sentences Γ is a subset of the set of sentences Δ, then the degree of joint acceptability of the members of Δ is less than or equal to the degree of joint acceptability of Γ.*

Proposition 1, however, is false. Suppose that a hypothesis H predicts all and only observations $o_1, ..., o_n$ occur for some $n > 1$. Then H receives maximal evidential support just when o_1 occurs and ... and o_n occurs, which is represented by sentences $O_1, ..., O_n$, respectively. Hence, it is more rational to accept a set Δ consisting of $H, O_1, ..., O_n$ than Δ without some observation statement O_i, since the set $\{O_1, ..., O_n\}$ is better support for H than any of its proper subsets. Let $\Gamma = \{H, O_1, ..., O_{n-1}\}$. Then $\Gamma \subseteq \Delta$ but the joint degree of acceptability of Δ is not less than the joint degree of acceptability of Γ.

The behavior of joint acceptability of sentences evoked by this example is common and reasonable. Ångström measured the wavelengths of four lines appearing in the emission spectrum of a hydrogen atom (410 nm, 434 nm, 486 nm, and 656 nm), from which J.J. Balmer noticed a regularity that fit the equation

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right), \text{when } n = 3, 4, 5, 6,$$

$R = 1.097 \times 10^7 m^{-1}$ is the *Rydberg* constant, and λ is the corresponding wavelength. From the observations that $\lambda = 410$ nm iff $n = 3$, $\lambda = 434$ nm iff $n = 4$, $\lambda = 486$ nm iff $n = 5$, and $\lambda = 656$ nm iff $n = 6$, Balmer's hypothesis—that this equation describes a series in emission spectra beyond these four visible values, that is for $n > 6$—is predicated on there being a series of measured wavelengths whose intervals are specified by $\frac{1}{\lambda}$. Later experiments confirmed over time that the Balmer series holds for values $n > 6$ through the ultraviolent spectrum. (It does not hold for all of the non-visible spectrum, however.) The point behind this historical example is that the grounds for a Balmer series describing the emission spectrum of a hydrogen atom *increased* as the set of confirmed values beyond Ångström's initial four measurements increased.

Returning to Morgan, the normative force of his impossibility result rests on a particular view about the structure of joint acceptability, and a commitment to viewing logic to be 1-formal. Morgan's method embraces 1-formality in the sense that he sees the subset principle, which is the logical property of monotonicity, as a constitutive norm for belief. What we have demonstrated in the discussion is that it is unreasonable to interpret the formal constraints of a belief structure to be normative constraints on joint acceptability. The reasons we marshaled against Morgan's view are not general principles about assertability or cognition, but instead are features of a concrete example that we understand and accept, but whose structure does not satisfy the prescriptions of belief structures. In doing so we found that belief structures are not a good formal model for joint acceptability. So, it is irrelevant that belief structures are monotone.

3.2 System P and Aggregation

Nevertheless we might wonder whether there is any interesting logical structure to non-monotonic logics. Even if Morgan's views on joint acceptability are mistaken, perhaps he's right about the broader point that non-monotonic logics fail to have enough structure to be properly classified as logics. To attack this question, first observe three standard properties that classical consequence $\vdash$ enjoys:

Reflexivity ($\alpha \vdash \alpha$),
Transitivity (If $\Gamma \vdash \delta$ for all $\delta \in \Delta$ and $\Delta \vdash \alpha$, then $\Gamma \vdash \alpha$), and
Monotonicity (If $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha$),

which were thinly disguised in Morgan's constraints on belief structures. What we want to investigate is whether these three properties are necessary to generate a non-trivial consequence relation. Or, more specifically, we wish to investigate whether there are any non-trivial consequence relations that do not satisfy the monotonicity condition.

Dov Gabbay [10] noticed that a restricted form of transitivity was helpful in isolating a class of non-monotonic consequence relations which nevertheless enjoy many properties of classical consequence. The result is important because it reveals that, *pace* Morgan, monotonicity isn't a fundamental property of consequence relations.

Gabbay observed that the general transitivity property of $\vdash$ is entailed by reflexivity, monotonicity, and a restricted version of transitivity he called *cumulative transitivity*:

Reflexivity ($\alpha \vdash \alpha$),
Cumulative Transitivity (If $\Gamma \vdash \delta$ for all $\delta \in \Delta$ and $\Gamma \cup \Delta \vdash \alpha$, then $\Gamma \vdash \alpha$), and
Monotonicity (If $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \alpha$).

This insight opened a way to systematically weaken the monotonicity property by exploring relations constructed from reflexivity and cumulative transitivity, which yields the class of cumulative non-monotonic logics [19].[4]

Since Gabbay's result, many semantics for non-monotonic logics and conditional logics have been found to share a core set of properties identified by Kraus, Lehmann, and Magidor [15], which they named axiom System P. System P is defined here by six properties of the consequence relation $\mid\!\sim$:

Reflexivity $\quad \alpha \mid\!\sim \alpha$

Left Logical Equivalence $\quad \dfrac{\models\ \alpha \leftrightarrow \beta;\quad \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$

[4] In light of these results we might return to Morgan and push the objection to his argument back to his notions of soundness and completeness. For example, we might propose to replace his definition of soundness by this: If $\Gamma \vdash_b A$, then $\Gamma \cup \{A\}$ *LE* $\Gamma \cup \{\neg A\}$, which would turn $\vdash_b$ into a KLM non-monotonic relation in terms of *LE*; or we might define $\Gamma \vdash_b A$ iff: for all $\in \mathbf{L}, \Gamma \cup \{A\}$ *LE* $\Gamma \cup \{\neg A\}$, which would yield a single, skeptical non-monotonic consequence operator in terms of *LE*. Thanks here to Hannes Leitgeb.

$$\textbf{Right Weakening} \qquad \frac{\models\ \alpha \rightarrow \beta;\quad \gamma \mid\!\sim \alpha}{\gamma \mid\!\sim \beta}$$

$$\textbf{And} \qquad \frac{\alpha \mid\!\sim \beta;\quad \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \wedge \gamma}$$

$$\textbf{Or} \qquad \frac{\alpha \mid\!\sim \gamma;\quad \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma}$$

$$\textbf{Cautious Monotonicity} \qquad \frac{\alpha \mid\!\sim \beta;\quad \alpha \mid\!\sim \gamma}{\alpha \wedge \beta \mid\!\sim \gamma}$$

System P is commonly regarded as the core set of properties that every non-monotonic consequence relation *should* satisfy [19, 11]. This assessment is based primarily on the observation that a wide range of non-monotonic logics have been found to satisfy this set of axioms,[5] including probabilistic semantics for conditional logics.[6] So, there is very little formal evidence for regarding belief structures as capturing minimal constraints on consequence relations.

However, some have drawn a stronger conclusion from the System P axioms than that non-monotonic logics are legitimate logical systems. Some authors have argued that System P marks minimal *normative* constraints on non-monotonic inference as such [19]. For instance, an impossibility argument may be extracted from Douven and Williamson [6] to the effect that no coherent probabilistic modeling of rational acceptance of a sentence α *can* be constructed on a logic satisfying axiom System P for probabilities assigned to α less than 1, and that any formal solution to the lottery paradox must have at least this much structure.[7] What Douven and Williamson are claiming is that a formal property on a probability space must be invariant under automorphisms, and they treat this formal property to have normative force in terms of 1-formality.

But it is one thing to assert that many non-monotonic logics are cumulative, or that it is formally desirable for non-monotonic logics to satisfy system P to preserve horn rules, say, or that there is no coherent and non-trivial Kolmogorov probability logic satisfying System P, and it is quite another matter to say that non-monotonic argument forms *should* satisfy System P, or to say that a logical account of rational acceptance *must* minimally satisfy the [**And**] rule of System P. One must be clear what class of argument structures are to be modeled, what are the most important properties of those structures to preserve in the system, whether those properties can be sensibly modeled, and whether those properties can be captured within the framework of choice. The problem with Fregean 1-formality is its end-run around these core questions.

[5] An important exception is Ray Reiter's default logic [28]. See Marek and Truszczynski [21] and Makinson [20] for textbook treatments of non-monotonic logic.

[6] See Judea Pearl [25, 26] which is developed around Adams' infinitesimal ϵ semantics, and Lehman and Magidor [15] which is built around non-standard probability.

[7] Although Douven and Williamson do not mention System P nor Gabbay's result, the weakened form of transitivity they discuss in their footnote 2 is cumulative transitivity, and the generality they gesture toward here suggests that they are discussing the class of cumulative non-monotonic logics.

For instance, there are good reasons to think that classical statistical inference forms do not satisfy the [**And**], [**Or**], and [**Cautious Monotonicity**] axioms of System P [16]. And there are logically interesting probabilistic logics that are weaker than System P. Among the weakest systems is System Y [34], which is built from 1-monotone capacities but nevertheless preserves *greatest lower bound* interval estimates on particular sequences of joins and meets of probabilistic events. This logic takes events whose marginal lower probability is known, but no information is known about the probabilistic relationship between each event, and preserves *glb* on a particular sequence of arbitrary meets and joins by inference rules, called *absorption rules*, for combining conjunctive and disjunctive events. The logic preserves bounds purely on the compositional structure of formulas, and does so without making independence assumptions among events whose joint measure is unknown.

An interesting feature of System P is that it weakens the link between monotonicity and demonstrative inference: [**Cautious Monotonicity**] and the [**And**] rule together specify the restricted conditions under which non-monotonically derived statements may be aggregated. But the aggregation properties of these two rules are not features of classical statistical inference forms, which are otherwise non-monotonic. This opens the way for studying sub-P systems of non-monotonic logics that accurately represent classical forms of statistical inference.

There are nevertheless three issues that sub-P logics face. The first concerns the syntactic capabilities one would like the logic to enjoy. This is to be contrasted to a model theoretic approach under which one constructs all possible combinations of sentences that are sound. One of the general challenges to formulating an adequate probabilistic logic, for example, is that any movement to introduce genuine logical connectives into the object language quickly erodes the precision of the logic to effect proofs of all sound combinations of formulas. An inductive logic enjoying minimally interesting modularity properties in the object language may not be complete.

Another issue facing sub-P logics is to specify to what degree the aggregation properties should be weakened. For instance, the KTW axioms [16] articulate a weakly aggregative logic that comes closest to axiomatizing evidential probability, although disjunction in this system is weaker than what one would expect from a standard probabilistic logic.

Finally, it should be noted that while sub-P logics may be more accurate for representing some epistemic concepts from a knowledge representation point of view, these logics are too weak on their own to be of much use inferentially. This is not an argument to favor interpreting System P as a normative constraint on non-monotonic reasoning. But it is a recognition that the 'given' structure of a problem domain—in this case, classical statistical inference forms—may be too weak to mirror the structure of an effective formal system. Nevertheless, it is crucial to be in a position to articulate what the given structure of a problem domain *is* in order to understand what must be added to, or assumed to hold within, an effective representation. Having these capabilities translates to being in a position to articulate precisely what assumptions are necessary to make in order to impose the structure of an effective model onto the problem domain. Having the capability to pinpoint assumptions of representation schemes is necessary for evaluating the conditions under which it is reasonable to make those assumptions.[8]

Here then we can see the power of the pure invariantist view. Logics should not be viewed as a collection of failed attempts to characterize logical consequence, nor should they be viewed to be an irreducible plurality. Rather, the multiplicity of logic systems and logical methods give us an ever increasing capacity for understanding how various formal languages and calculi work, the relationships these systems bear to each another, and a catalogue of what cannot be done and why.

The question of why one logic rather than another is answered in the end by the use of a logic. The contribution that logic makes to giving an answer this question is a specification of the structural, inferential, conceptual, and complexity properties of specific systems, or classes of systems. In other words, a logic will provide the specifications of its properties and tell you how it is related to other known logics, and even to other pieces of mathematics. Another component to answering this question is a judgment of how well the salient features of the problem you'd like to represent—the argument, the model to check, the agent playing a game—are represented by a logical system. Only when we learn enough about those types of problems—arguments, models, multi-agent games—are we able to sort them into classes. Likewise, our investigation of logical calculi can reveal places to look that we otherwise may not have considered—just as we find in other branches of mathematics.

4 Conclusion

Formal solutions to applied logic problems are influenced by the relationship between the formal language of the proposed framework, the structure imposed by the problem, and the purpose of the formal model. Simply because a formal system has a certain set of attractive mathematical or computational properties does not entail that the domain it is intended to model should have that structure. This is the error that 1-formality seduces us to make, and it is the underlying reason why Fregean psychologism should be rejected. Applying logic involves assessing the structure of the problem domain, understanding the structure of the logic, then reconciling the differences between these two structures with a clear specification of the task one has in mind for the formal representation.

Solutions to applied logic problems—including philosophical logic problems—take the form of optimization searches. Each candidate solution picks salient properties of the problem to represent and attempts to capture these features in the formal framework while balancing precision against usefulness. Only by studying various candidate solutions, various attempts to strike this balance,

[8] Other work on sub-P logics include [31],[1], [14].

will we then be in a position to recognize an acceptable formal solution to the problem at hand.

Acknowledgments: This essay is based upon work supported by FCT award SFRH/BPD-13699-2003 and *The Leverhulme Trust*. Thanks to Hannes Leitgeb and an anonymous referee for their comments on an earlier draft.

References

1. Horacio Arló-Costa. Non-adjunctive inference and classical modalities. *Journal of Philosophical Logic*, 34:581–605, 2005.
2. J. C. Beall and Greg Restall. *Logical Pluralism*. Oxford University Press, Oxford, 2006.
3. Michael Byrd. Russell, logicisn and the choice of logical constants. *Notre Dame Journal of Formal Logic*, 30(3):343–361, 1989.
4. Enrique Casanovas. Logical operations and invariance. *Journal of Philosophical Logic*, 36(1):33–60, 2007.
5. Brian Chellas. *Modal Logic*. Cambridge University Press, Cambridge, 1980.
6. Igor Douven and Timothy Williamson. Generalizing the lottery paradox. *The British Journal for the Philosophy of Science Philosophy of Science*, 57(4):755–779, 2006.
7. Ronald A. Fisher. Uncertain inference. *Proceedings of the American Academy of Arts and Sciences*, 71:245–258, 1936.
8. Ronald A. Fisher. On the mathematical foundations of theoretical statistics. In J. H. Bennett, editor, *Collected Papers of R. A. Fisher*. University of Adelaide, Adelaide, 1971. Originally published in 1922.
9. Gottlob Frege. *The Foundations of Arithmetic*. Basil Blackwell, Oxford, j. l. austin (trans). edition, 1893(1950).
10. Dov Gabbay. Theoretical foundations for nonmonotonic reasoning in expert systems. In K. Apt, editor, *Logics and Models of Concurrent Systems*. Springer-Verlag, Berlin, 1985.
11. Joseph Y. Halpern. *Reasoning about Uncertainty*. MIT Press, Cambridge, MA, 2003.
12. Gilbert Harman. *Change in View*. MIT Press, Cambridge, MA, 1986.
13. Gilbert Harman and Sanjeev R. Kulkarni. The problem of induction. *Philosophy and Phenomenological Research*, 2005. Presented at the 2005 Rutgers Epistemology Workshop.
14. James Hawthorne and David C. Makinson. The quantitative/qualitative watershed for rules of uncertain inference. *Studia Logica*, 2007.
15. Sarit Kraus, Daniel Lehman, and Menachem Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44:167–207, 1990.
16. Henry E. Kyburg, Jr., Choh Man Teng, and Gregory Wheeler. Conditionals and consequences. *Journal of Applied Logic*, 2007.
17. Adolf Lindenbaum and Alfred Tarski. On the limitations of the means of expression of deductive theories. In John Corcoran, editor, *Logic, Semantics, Metamathematics*. Hackett, Indianapolis, 1983.
18. John MacFarlane. *What does it mean to say that logic is formal*. PhD thesis, University of Pittsburgh, 2000.

19. David C. Makinson. General patterns in nonmonotonic reasoning. In Dov Gabbay, Chris Hogger, and John Alan Robinson, editors, *Handbook of Logic in Artificial Intelligence and Uncertain Reasoning*, volume 3. Clarendon Press, Oxford, 1994.
20. David C. Makinson. *Bridges from Classical to Nonmonotonic Logic*. King's College Publications, London, 2005.
21. V. Wiktor Marek and Mirosław Truszczyński. *Nonmonotonic Logic*. Springer-Verlag, Berlin, 1993.
22. Van McGee. Logical operations. *Journal of Philosophical Logic*, 25:567–580, 1996.
23. Charles G. Morgan. Non-monotonic logic is impossible. *Canadian Journal of Artificial Intelligence*, 42:19–25, 1998.
24. Charles G. Morgan. The nature of nonmonotonic reasoning. *Minds and Machines*, 10:321–360, 2000.
25. Judea Pearl. *Probabilistic Reasoning in Intelligent Systems*. Morgan Kaufmann, San Francisco, CA, 1988.
26. Judea Pearl. System Z: A natural ordering of defaults with tractable applications to default reasoning. In *Theoretical Aspects of Reasoning about Knowedge*, pages 121–135, 1990.
27. Ian Pratt-Hartmann. Fragments of language. *Logic, Language and Information*, 13(2):207–223, 2004.
28. Ray Reiter. A logic for default reasoning. *Artificial Intelligence*, 13:81–132, 1980.
29. Greg Restall. Carnap's tolerance, meaning and logical pluralism. *Journal of Philosophy*, 99:426–443, 2002.
30. Gila Sher. A characterization of logical constants is possible. *Theoria*, 18(47):189–198, 2003.
31. Allard Tamminga. *Belief Dynamics: Epistemological Investigations*. PhD thesis, University of Amsterdam, 2001.
32. Johan van Benthem. Logical constants across varying types. *Notre Dame Journal of Formal Logic*, 30(3):315–342, 1989.
33. Ken Warmbrod. Logical constants. *Mind*, 108(431):503–538, 1999.
34. Gregory Wheeler. Rational acceptance and conjunctive/disjunctive absorption. *Journal of Logic, Language and Information*, 15(1-2):49–63, 2006.