Rational Acceptance and Conjunctive/Disjunctive Absorption

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Abstract. A bounded formula is a pair consisting of a propositional formula ϕ in the first coordinate and a real number within the unit interval in the second coordinate, interpreted to express the lower-bound probability of ϕ. Converting conjunctive/disjunctive combinations of bounded formulas to a single bounded formula consisting of the conjunction/disjunction of the propositions occurring in the collection along with a newly calculated lower probability is called *absorption*. This paper introduces two inference rules for effecting conjunctive and disjunctive absorption and compares the resulting logical system, called System Y, to axiom System P. Finally, we demonstrate how absorption resolves the lottery paradox and the paradox of the preference.

Keywords: Probabilistic logic, rational acceptance, the lottery paradox, System P, bounded uncertain reasoning.

1. Introduction

The *lottery paradox* (Kyburg 1961) arises from considering a fair 1000 ticket lottery that has exactly one winning ticket. If this much is known about the execution of the lottery it is therefore rational to accept that one ticket will win. Suppose that an event is very likely if the probability of its occurring is greater than 0.99. On these grounds it is rational to accept the proposition that ticket 1 of the lottery will not win. Since the lottery is fair, it is rational to accept that ticket 2 won't win either—indeed, it is rational to accept for any individual ticket i of the lottery that ticket i will not win. However, accepting that ticket 1 won't win, accepting that ticket 2 won't win, . . . , and accepting that ticket 1000 won't win seems to entail that it is rational to accept that no ticket will win, which entails that it is rational to accept the contradictory proposition that one ticket will win and no ticket will win.

The *paradox of the preface* (Makinson 1965) arises from considering an earnest and careful author who writes a preface for a book he has just completed. For each page of the book, the author believes that it is without error. Yet in writing the preface the author believes that there is surely a mistake in the book, somewhere, so offers an apology to his readers. Hence, if the author conjoins his individual beliefs, he appears to be committed to both the claim that every page of his book is without error and the claim that at least one page contains an error.

Abstracted from their particulars, the lottery paradox and the paradox of the preface are each designed to demonstrate that three attractive principles for governing rational acceptance lead to contradiction, namely that

1. It is rational to accept a proposition that is very likely true,
2. It is not rational to accept a proposition that you are aware is inconsistent, and
3. If it is rational to accept a proposition A and it is rational to accept another proposition A', then it is rational to accept $A \wedge A'$

are jointly inconsistent. For this reason, these two paradoxes are sometimes referred to as *the paradoxes of rational acceptance.*

In (Wheeler 2005) I advocated that we adopt the *structural view of rational acceptance* to resolve these two paradoxes. The structural view is motivated by observing that the problem raised by the paradoxes of rational acceptance is a general one of how to reconcile the first and third legislative principles. But to study the general relationship between rational acceptance and logical consequence, we need to understand valid *forms* of arguments whose premises are rationally accepted propositions. This point suggests three conditions for us to observe. First, it is important to define the notion of rational acceptance independently of any particular language, since this notion is serving as a semantic property that is thought to be preserved (in a restricted sense) under entailment. Second, to formally represent an argument composed of rationally accepted propositions we must have facilities for formally representing their combination within an object language. Finally, of formal languages that satisfy the first two properties, preference should be given to those within systems that make the relationship between rational acceptance and logical consequence transparent.

So what kind of proposal counts as a structured proposal? The short answer is that the structural constraints are intended to narrow our consideration to just those proposals that feature a genuine probabilistic logic. In this essay I present an outline of a candidate solution to the paradoxes of rational acceptance that I favor. This proposal is constructed around two rules called *conjunctive absorption* (CA) and *disjunctive absorption* (DA) that are used to convert conjunctions/disjunctions of rationally accepted propositions to rationally accepted conjunctions/disjunctions. The paper first introduces these two rules and then discusses some properties that they enjoy. Finally, I discuss how to apply absorption to resolve the lottery and preface paradoxes.

The key to progress on either of these two paradoxes is to understand the behavior of operators roughly similar to Boolean disjunction and conjunction but that include a provision for the possible depletion of probability mass when probabilistic events are combined. Just as propositional calculi may be viewed as studies of some class or other of propositional connectives, a calculus for rationally accepted sentences may be viewed as the study of connectives for rationally accepted sentences. The structural view of the paradoxes of rational acceptance simply holds that this is the right project to undertake if one's aim is to resolve these paradoxes.

2. Combined Events and their Measure

In this section we define a language of *bounded formulas* that expresses basic propositions and their associated levels of confidence. This paper studies the interpretation of a bounded formula $\langle \phi, e \rangle$ as the *inner-measure* (Halmos 1950) e induced by a classical probability measure μ of the propositional formula ϕ. The next two subsections define this notion.

2.1. Syntax and Semantics

Let $\Phi = \{p, q, p_1, p_2, \ldots\}$ be an infinite set of primitive propositions, and $\neg$ and $\vee$ be the primitive Boolean connectives. The logical connectives $\wedge, \rightarrow$, and $\leftrightarrow$ are the derived connectives, hence $\rightarrow$ is the material conditional. Let $\top$ stand for $p \vee \neg p$, and $\bot$ stand for $\neg\top$. The set of *propositional formulas* is the set Φ closed under the primitive Boolean connectives $\neg$ and $\vee$. Whereas p and q stand for primitive propositions, ϕ and ψ stand for propositional formulas.

Let Υ be an infinite set of expression of the form $\langle \phi, e \rangle$, called *basic bounded formulas*, where ϕ is a propositional formula and $0 \leq e \leq 1$. The basic bounded formula $\langle p, e \rangle$ expresses that "the probability that p is no less than e". *Complex bounded formulas* are the set Υ closed under $\{\vee, \wedge\}$. There are no negated nor conditional bounded formulas: that is, neither $\neg\langle \phi, e \rangle$ nor $\langle \phi, e \rangle \rightarrow \langle \phi', e' \rangle$ appear in the language. We denote the set of basic and complex bounded formulas by Υ^+.

Semantics for basic bounded formulas are provided in terms of a *probability space* $(W, \mathcal{F}, \mu)$, where $\mathcal{F}$ is a σ-algebra over a set W and

$\mu : \mathcal{F} \longrightarrow [0, 1]$ is a probability measure defined on the space $(W, \mathcal{F}, \mu)$ satisfying

P1. $\mu(W) = 1$

P2. $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=i}^{\infty}(A_i)$, when A_i are countable, pairwise disjoint elements of $\mathcal{F}$.

When A and B are disjoint members of $\mathcal{F}$ and W is finite, a special case of P2 is

P2* $\mu(A \cup B) = \mu(A) + \mu(B)$,

from which, along with P1, a useful general additivity property may be derived, namely

P2′ $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$, when A and B are elements of $\mathcal{F}$.

A *probability structure* is a tuple $M = (W, \mathcal{F}, \mu, \pi)$, where $(W, \mathcal{F}, \mu)$ is a probability space and π is an interpretation function associating each element (world) $w \in W$ with a truth assignment on the primitive propositions in Φ such that $\pi(w)(p) \in \{true,\ false\}$ for each $w \in W$ and for every $p \in \Phi$.

Assume now that $\mathcal{F}$ is a subalgebra of an algebra $\mathcal{F}'$, $\mathcal{F} \subseteq \mathcal{F}'$. Observe that if μ is a probability measure on $\mathcal{F}$ and $A \in \mathcal{F}' - \mathcal{F}$, then $\mu(A)$ is not defined since A is not in the domain of μ. However, we may extend the measure μ to the set A by defining inner and outer measures (Halmos 1950) to represent our uncertainty with respect to the precise measure of A, an approach developed in (Fagin, *et. al.* 1990) and studied by (Walley 1991) and (Halpern 2003).

First, define an extension of a probability space as follows.

DEFINITION 1. *A probability space* $(W, \mathcal{F}', \mu')$ *is an extension of* $(W, \mathcal{F}, \mu)$ *if* $\mathcal{F}' \supseteq \mathcal{F}$ *and* $\mu'(A) = \mu(A)$ *for all* $A \in \mathcal{F}$. *If* $(W, \mathcal{F}', \mu')$ *is an extension of* $(W, \mathcal{F}, \mu)$, *then* μ' *is said to extend* μ.

We may then define inner and outer measures.

DEFINITION 2. *Let* $\mathcal{F}$ *be a subalgebra of an algebra* $\mathcal{F}'$, $\mu : \mathcal{F} \longrightarrow [0, 1]$ *a probability measure defined on the space* $(W, \mathcal{F}, \mu)$, *and* A *an arbitrary set in* $\mathcal{F}' - \mathcal{F}$. *Then define the* inner measure μ_* induced by μ *and the* outer measure μ^* induced by μ *as:*

$\mu_*(A) = \sup\{\mu(B) : B \subseteq A, B \in \mathcal{F}\}$ (inner measure of A)*;*

$\mu^*(A) = \inf\{\mu(B) : B \supseteq A, B \in \mathcal{F}\}$ (outer measure of A).

We now observe some properties of inner and outer measures:

P3. $\mu_*(A \cup B) \geq \mu_*(A) + \mu_*(B)$, when A and B are disjoint (superadditivity);

P4. $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$, when A and B are disjoint (subadditivity);

P5. $\mu_*(A) = 1 - \mu^*(\overline{A})$;

P6. $\mu_*(A) = \mu^*(A) = \mu(A)$, if $A \in \mathcal{F}$.

Properties P3 and P4 follow from P1 and P2. Note that when W is finite, P3 and P4 follow from P1 and P2*. P5 makes explicit the relationship between inner and outer measures. By P2, for each set A, there are measurable sets $B, C \in \mathcal{F}$ such that $B \subseteq A \subseteq C$ and $\mu_*(A) = \mu(B)$ and $\mu^*(A) = \mu(C)$. Note then the limit cases: if there are no measurable sets containing A other than the entire space W, then $\mu^* = 1$; if there are no nonempty measurable sets contained in A, then $\mu_*(A) = 0$. P6 makes explicit that inner and outer measures strictly extend μ: if an event A is measurable, then the inner (outer) measure of A is $\mu(A)$.

Also by P2 and P1, we may generalize P3 to

P3′. $\mu_*(A \cup B) \geq \mu_*(A) + \mu_*(B) - \mu_*(A \cap B)$ (generalized superadditivity).

In short, inner and outer measures may be viewed as a method to offer the best estimate for a set A based upon known measures of sets containing or contained by A. A result, proved in (Ruspini 1987), makes this notion precise.

THEOREM 1. *(i) If* $(W, \mathcal{F}', \mu')$ *is an extension of* $(W, \mathcal{F}, \mu)$ *and* $A \in \mathcal{F}'$*, then* $\mu_*(A) \leq \mu'(A) \leq \mu^*(A)$. *(ii) There are extensions* $(W, \mathcal{F}', \mu')$ *and* $(W, \mathcal{F}'', \mu'')$ *of* $(W, \mathcal{F}, \mu)$ *such that* $A \in \mathcal{F}'$, $A \in \mathcal{F}''$, $\mu'(A) = \mu_*(A)$ *and* $\mu''(A) = \mu^*(A)$.

Suppose that $M = (W, \mathcal{F}, \mu, \pi)$ is a probability structure and that $\mathcal{G}$, called a *basis of* M, is a set of non-empty, disjoint subsets of W such that $\mathcal{F}$ consists precisely of all countable unions of members of $\mathcal{G}$. When $\mathcal{F}$ is finite, then there is a unique basis $\mathcal{G}$ consisting precisely of the minimal elements of $\mathcal{F}$ (Fagin, *et. al.* 1990). Given the probability of every set in the basis set $\mathcal{G}$, the probability of every measurable set is calculated by $P2$. Furthermore, the inner and outer measures may also be defined in terms of the basis $\mathcal{G}$: the inner measure of event A is the sum of the measures of the basis set elements that are subsets of A, whereas the outer measure of event A is the sum of the measure of the basis elements that intersect A (Fagin, *et. al.* 1990).

Define for primitive proposition $p \in \Phi$, $M, w \models p$ iff $\pi(w)(p) = true$. Then proceed by induction on the structure of propositional formulas.

$M \models \phi \wedge \phi'$ iff $M \models \phi$ and $M \models \phi'$;

$M \models \phi \vee \phi'$ iff $M \models \phi$ or $M \models \phi'$;

$M \models \neg\phi$ iff $M \not\models \phi$;

In the next sections we discuss how to extend the definition for propositional formulas to bounded formulas.

2.2. Conjunction and disjunction for bounded formulas

Since μ is defined on events rather than propositions, let $[\![p]\!]_M$ denote the set of worlds within W in M where p is true. The following proposition makes explicit the relationship between propositions and events.

PROPOSITION 1. *For arbitrary propositional formulas ϕ and ϕ',*

i. $[\![\phi \wedge \phi']\!]_M = [\![\phi]\!]_M \cap [\![\phi']\!]_M$,

ii. $[\![\phi \vee \phi']\!]_M = [\![\phi]\!]_M \cup [\![\phi']\!]_M$,

iii. $[\![\neg\phi]\!]_M = \overline{[\![\phi]\!]_M}$.

The bounded formula $\langle \phi, e \rangle$ is interpreted as the inner measure of the set $[\![\phi]\!]_M$. We may extend the definition for propositional formulas to include Boolean conjunction and disjunction of bounded formulas in the following manner:

$M \models \langle \phi_1, e_1 \rangle \vee \ldots \vee \langle \phi_n, e_n \rangle$ iff $\mu_*([\![\phi_1]\!]_M) = e_1$ or $\ldots$ or $\mu_*([\![\phi_n]\!]_M) = e_n$;

$M \models \langle \phi_1, e_1 \rangle \wedge \ldots \wedge \langle \phi_n, e_n \rangle$ iff $\mu_*([\![\phi_1]\!]_M) = e_1$ and $\ldots$ and $\mu_*([\![\phi_n]\!]_M) = e_n$.

However, Boolean combinations of bounded formulas are not very informative: each bounded formula is treated as a proposition that expresses its own lower-bound probability. The probability that p is at least .999 and the probability that p' is at least .999 is satisfied by M just in case the inner measure of p is .999 and the inner measure of p' is .999, which does not entail that the probability of the conjunction $p \wedge p'$ is at least .999. There is an important difference between the probability of a conjunctive (disjunctive) event and the conjunction (disjunction) of probabilistic events that prohibits distributing Boolean $\wedge$ and $\vee$ freely over bounded formulas.

Nevertheless, if individual probabilistic events have a lower-bound probability then one may determine a lower bound on conjunctive and also disjunctive combinations of those events based upon the properties of inner measures. We do this by first considering bounds generated by the properties of the classical probability measure μ.

LEMMA 1. *For arbitrary propositions p and q and a probability structure M, if $\mu([\![p]\!]_M)$ and $\mu([\![q]\!]_M)$ are defined, then*

(i) $\mu([\![p]\!]_M \cap [\![q]\!]_M)$ *lies within the interval* $[\max(0, \mu([\![p]\!]_M) + \mu([\![q]\!]_M) - 1), \min(\mu([\![p]\!]_M), \mu([\![q]\!]_M)]$, *and*

(ii) $\mu([\![p]\!]_M \cup [\![q]\!]_M)$ *lies within the interval* $[\max(\mu([\![p]\!]_M), \mu([\![q]\!]_M)), \min(\mu([\![p]\!]_M) + \mu([\![q]\!]_M), 1)]$.

Proof. (i). By P2$'$, $\mu([\![p]\!]_M \cap [\![q]\!]_M) = \mu([\![p]\!]_M) + \mu([\![q]\!]_M) - \mu([\![p]\!]_M) \cup [\![q]\!]_M) \geq \mu([\![p]\!]_M) + \mu([\![q]\!]_M) - 1$. Hence, $\mu([\![p]\!]_M \cap [\![q]\!]_M) \geq \max(0, \mu([\![p]\!]_M) + \mu([\![q]\!]_M) - 1)$. Next, observe that $\mu([\![p]\!]_M) = \mu(\{[\![p]\!]_M \cap [\![q]\!]_M\} \cup \{[\![p]\!]_M \cap \underline{[\![q]\!]_M}\})$ and $\mu([\![p]\!]_M) = \mu([\![p]\!]_M \cap [\![q]\!]_M) + \mu([\![p]\!] \cap \overline{[\![q]\!]_M}) \geq \mu([\![p]\!]_M) \cap [\![q]\!]_M)$. Similarly for $\mu([\![q]\!]_M)$. So $\mu([\![p]\!]_M \cap [\![q]\!]_M) \leq \min(\mu([\![p]\!]_M), \mu([\![q]\!]_M))$.

(ii) From the proof of (i), we have that $\mu([\![p]\!]_M) \geq \mu([\![p]\!]_M \cap [\![q]\!]_M)$ and $\mu([\![q]\!]_M) \geq \mu([\![p]\!]_M \cap [\![q]\!]_M)$. Then from P2$'$, $\mu([\![p]\!]_M \cup [\![q]\!]_M) = \mu([\![p]\!]_M) + \mu([\![q]\!]_M) - \mu([\![p]\!]_M \cap [\![q]\!]_M) \geq \mu([\![p]\!])$ and $\mu([\![p]\!]_M \cup [\![q]\!]_M) = \mu([\![p]\!]_M) + \mu([\![q]\!]_M) - \mu([\![p]\!]_M \cap [\![q]\!]_M) \geq \mu([\![q]\!])$. So, $\mu([\![p]\!]_M \cup [\![q]\!]_M) \geq \max(\mu([\![p]\!]_M), \mu([\![q]\!]_M))$. Finally, $\mu([\![p]\!]_M \cup [\![q]\!]_M) = \mu([\![p]\!]_M) + \mu([\![q]\!]_M) - \mu([\![p]\!]_M) \cap [\![q]\!]_M) \leq \mu([\![p]\!]_M) + \mu([\![q]\!]_M)$ and $\mu([\![p]\!]_M \cup [\![q]\!]_M) \leq 1$. So, $\mu([\![p]\!]_M \cup [\![q]\!]_M) \leq \min(\mu([\![p]\!]_M) + \mu([\![q]\!]_M), 1)$.

Lemma 1(i) corresponds to the *ignorance p-strategy* discussed in (Dekhryar and Subrahmanian 2000), where the probability of $p \wedge q$ is bounded below by $\mu([\![p]\!]_M) + \mu([\![q]\!]_M) - 1$ when this expression is positive, 0 otherwise, and the probability of $p \wedge q$ is bounded above by the lower of the two marginal probabilities, $\mu([\![p]\!]_M)$ or $\mu([\![q]\!]_M)$.

An important property of Lemma 1 is that the bounds are determined purely by the properties of the measure μ; there is no probabilistic assumption made about the events $[\![p]\!]_M$ and $[\![q]\!]_M$ other than that each is measurable, i.e., that each set is in $\mathcal{F}$. The point we wish to stress is that there are no probabilistic assumptions made about how one event is related to the other. Indeed, Lemma 1(i) defines the *lub* and *glb* for $\mu(\llbracket p \rrbracket_M \cap \llbracket q \rrbracket_M)$ when no assumption is made about the relationship between $\llbracket p \rrbracket_M$ and $\llbracket q \rrbracket_M$.

We now extend this result to inner measures.

THEOREM 2. *For arbitrary propositions p and q and a probability structure M, if $\mu_* \llbracket p \rrbracket_M$ and $\mu_* \llbracket q \rrbracket_M$ are defined, then*

(i) $\mu_*(\llbracket p \rrbracket_M \cap \llbracket q \rrbracket_M)$ *lies within the interval*
$[\max(0, \mu_*(\llbracket p \rrbracket_M) + \mu_*(\llbracket q \rrbracket_M) - 1), \min(\mu_*(\llbracket p \rrbracket_M), \mu_*(\llbracket q \rrbracket_M)].$

(ii.) $\mu_*(\llbracket p \rrbracket_M \cup \llbracket q \rrbracket_M)$ *lies within the interval*
$[\max(\mu_*(\llbracket p \rrbracket_M), \mu_*(\llbracket q \rrbracket_M)), \min(\mu_*(\llbracket p \rrbracket_M) + \mu_*(\llbracket q \rrbracket_M), 1)].$

Proof. Observe that the statement is equivalent to Lemma 1 by P6 when $\llbracket p \rrbracket_M$ and $\llbracket q \rrbracket_M$ are measurable, i.e. when $\llbracket p \rrbracket_M$ and $\llbracket q \rrbracket_M$ are in $\mathcal{F}$.

By P3′, $\mu_*(\llbracket p \rrbracket_M \cap \llbracket q \rrbracket_M) \geq \mu_*(\llbracket p \rrbracket_M) + \mu_*(\llbracket q \rrbracket_M) - \mu_*(\llbracket p \rrbracket_M) \cup \llbracket q \rrbracket_M) \geq \mu_*(\llbracket p \rrbracket_M) + \mu_*(\llbracket q \rrbracket_M) - 1$. Hence, $\mu_*(\llbracket p \rrbracket_M \cap \llbracket q \rrbracket_M) \geq \max(0, \mu_*(\llbracket p \rrbracket_M) + \mu_*(\llbracket q \rrbracket_M) - 1)$.

From P3, $\mu_*(\llbracket p \rrbracket_M) = \mu_*(\{\llbracket p \rrbracket_M \cap \llbracket q \rrbracket_M\} \cup \{\llbracket p \rrbracket_M \cap \overline{\llbracket q \rrbracket_M}\}) \geq \mu_*(\llbracket p \rrbracket_M \cap \llbracket q \rrbracket_M) + \mu_*(\llbracket p \rrbracket \cap \overline{\llbracket q \rrbracket_M}]) \geq \mu_*(\llbracket p \rrbracket_M \cap \llbracket q \rrbracket_M)$. Similarly for $\mu_*(\llbracket q \rrbracket_M)$. Then from P3′, we get $\mu_*(\llbracket p \rrbracket_M \cup \llbracket q \rrbracket_M) \geq \mu_*(\llbracket p \rrbracket_M) + \mu_*(\llbracket q \rrbracket_M) - \mu_*(\llbracket p \rrbracket_M \cap \llbracket q \rrbracket_M) \geq \mu_*(\llbracket p \rrbracket_M)$, and likewise $\mu_*(\llbracket p \rrbracket_M \cup \llbracket q \rrbracket_M) \geq \mu_*(\llbracket q \rrbracket_M)$. So $\mu_*(\llbracket p \rrbracket_M \cup \llbracket q \rrbracket_M) \geq \max(\mu_*(\llbracket p \rrbracket_M), \mu_*(\llbracket q \rrbracket_M))$.

Theorem 2(i) says that given an inner measure for p (induced by μ under M) and an inner measure for q, the inner measure for $p \wedge q$ is bounded from below by $\mu_*(\llbracket p \rrbracket_M) + \mu_*(\llbracket q \rrbracket_M) - 1$ when this expression is positive, 0 otherwise, and the inner measure of $p \wedge q$ is bounded above by the lower of the two marginal inner measures, $\mu_*(\llbracket p \rrbracket_M)$ or $\mu_*(\llbracket q \rrbracket_M)$. Part (ii) says that given an inner measure for p and an inner measure for q, *both induced by μ under M*, the inner measure for $p \vee q$ is bounded from below by the larger of the two marginal inner measures, $\mu_*(\llbracket p \rrbracket_M)$ or $\mu_*(\llbracket q \rrbracket_M)$, and bounded from above by $\mu_*(\llbracket p \rrbracket_M) + \mu_*(\llbracket q \rrbracket_M)$ when this expression is less than 1, and bounded by 1 otherwise.

Viewed in terms of bounded formulas, Theorem 2 states that $\langle p, e \rangle$ and $\langle q, e' \rangle$ entail that $\langle p \wedge q, \max(0, e + e' - 1) \rangle$ and that $\langle p, e \rangle$ or $\langle q, e' \rangle$ entail that $\langle p \vee q, \max(e, e') \rangle$.

We may generalize this result as follows.

THEOREM 3. *For arbitrary bounded formulas* $\langle \phi_1, e_1 \rangle, \ldots, \langle \phi_1, e_1 \rangle$,

(i) *if* $M \models \langle \phi_1, e_1 \rangle \wedge \ldots \wedge \langle \phi_n, e_n \rangle$ *then* $M \models \langle \phi_1 \wedge \ldots \wedge \phi_n, \max(0, (\sum_{i=1}^{n} e_i - (n-1)) \rangle$;

(ii) *if* $M \models \langle \phi_1, e_1 \rangle \vee \ldots \vee \langle \phi_n, e_n \rangle$ *then* $M \models \langle \phi_1 \vee \ldots \vee \phi_n, \max(e_1, \ldots, e_n) \rangle$.

Proof. The proof follows directly from the definition of bounded formulas and Theorem 2.

Our principle interest is in preserving the *glb* of bounded formulas closed under logical consequence. We remark then that Theorem(ii) is perhaps a stronger result than one might otherwise expect. For instance, it might be thought that the *glb* of $\mu(\llbracket p \rrbracket_M \cup \llbracket q \rrbracket_M)$ should be $[\min(\mu(\llbracket p \rrbracket_M), \mu(\llbracket q \rrbracket_M))$ on the grounds that assuming the max value allows for the possibility to absorb a disjunction bounded by the max-value e disjunct and also to accept the negation of that max-valued disjunct, whereat one may derive, by disjunctive elimination, the remaining disjunct that is strictly less than e. However, this line of reasoning is blocked when working with inner-measures. This may point to a limitation in *the application of* inner-measures to model rational acceptance and (uncertain) reasoning by cases. Compare (Kyburg, Teng and Wheeler, forthcoming).

COROLLARY 1. *For arbitrary bounded formulas* $\langle \phi, e \rangle$ *and* $\langle \psi, e_2 \rangle$,

(i) $\langle \phi, e_1 \rangle \models_M \langle \psi, e_2 \rangle$ *iff* $\phi \models_M \psi$ *and* $e_2 \leq e_1$.

(ii) $\langle \phi, e_1 \rangle \equiv \langle \psi_2, e_2 \rangle$ *iff* $\langle \phi, e_1 \rangle_M \dashv\models_M \langle \phi, e_2 \rangle$ *iff* $\phi_1 \equiv \phi_2$ *and* $e_1 = e_2$.

Proof. Note that the bound of any sub-formula of ϕ does not exceed e_1, by induction on the structure of ϕ. Observe then that (i) follows from the 0-ary conjunction and 0-ary disjunction $\langle \phi_1, e_1 \rangle \wedge \emptyset \equiv \langle \phi_1, e_1 \rangle$ and $\langle \phi_1, e_1 \rangle \vee \emptyset \equiv \langle \phi_1, e_1 \rangle$, respectively, since $\max(e_1) = e_1$. Clause (ii) then follows trivially from (i).

Next, we observe that conjunctions (disjunctions) of bounded formulas *commute* and *associate*, and that error-bounds are weakly, positively *monotone*.

PROPOSITION 2. *Define* $\circ \in \{\vee, \wedge\}$. *Then for arbitrary* $\langle \phi_1, e_1 \rangle \circ \cdots \circ \langle \phi_n, e_n \rangle$, *let* $e_A = \max(e_1, \ldots, e_n)$ *when* $\circ = \vee$, *and let* $e_A = \max(0, \sum_{i=1}^{n} -(n-1))$ *when* $\circ = \wedge$. *Then the following properties hold:*

– *if* $(((\langle\phi_1, e_1\rangle \circ \langle\phi_2, e_2\rangle) \circ \langle\phi_3, e_3\rangle) \models_M \langle\phi_1 \circ \phi_2 \circ \phi_3, e_A\rangle$ *then* $(\langle\phi_1, e_1\rangle \circ (\langle\phi_2, e_2\rangle \circ \langle\phi_3, e_3\rangle)) \models_M \langle\phi_1 \circ \phi_2 \circ \phi_3, e_A\rangle$ *(Associativity)*

– *If* $\langle\phi_1, e_1\rangle \circ \langle\phi_2, e_2\rangle \models_M \langle\phi_1 \circ \phi_2, e_A\rangle$ *then* $M \models \langle\phi_2, e_2\rangle \circ \langle\phi_1, e_1\rangle \models_M \langle\phi_1 \circ \phi_2, e_A\rangle$ *(Commutativity)*

– *if* $M \models \langle\phi_1, e_1\rangle \circ \langle\phi_2, e_2\rangle \models_M \langle\phi_1 \circ \phi_2, e_A\rangle$ *then* $M \models \langle\phi_1, e_1\rangle \circ \langle\phi_2, e_2\rangle \circ \langle\phi_3, e_3\rangle \models_M \langle\phi_1 \circ \phi_2, e_A\rangle$ *(Weak Monotonicity)*

We now propose two inference rule schemata called *conjunction absorption* (CA) and *disjunction absorption* (DA).

$$\frac{\langle\phi_1, e_1\rangle \wedge \ldots \wedge \langle\phi_n, e_n\rangle}{\langle\phi_1 \wedge \ldots \wedge \phi_n, \max(0, (\sum_{i=1}^{n} e_i - (n-1))\rangle} \quad \text{CA}$$

$$\frac{\langle\phi_1, e_1\rangle \vee \ldots \vee \langle\phi_n, e_n\rangle}{\langle\phi_1 \vee \ldots \vee \phi_n, \max(e_1, ..., e_n)\rangle} \quad \text{DA}$$

The soundness of each rule is immediate from Theorem 3. CA and DA provide sound inference rules to convert collections of bounded formulas into a single bounded formula. Thus, they provide a means to consolidate a collection of propositions, for which there are particular fixed lower-bound probability, to either a single disjunctive proposition with a fixed lower-bound probability or a single conjunctive proposition with a fixed lower-bound probability.

2.3. Absorption and System Y

Call the logic resulting from the language Υ^+ with the rules CA and DA *System Y*. We now identify some properties of System Y.

DEFINITION 3 (Absorption $\Vdash_A$). *Define* $\langle\phi_1, e_1\rangle \circ \cdots \circ \langle\phi_n, e_n\rangle \Vdash_A \langle\phi_1 \circ \cdots \circ \phi_n, e_A\rangle$ *as an instance of either CA or DA, where* $\circ \in \{\wedge, \vee\}$, $\langle\phi_1, e_1\rangle \circ \cdots \circ \langle\phi_n, e_n\rangle$ *are premises,* $\langle\phi_1 \circ \cdots \circ \phi_n, e_A\rangle$ *is the conclusion, and* e_A *is the calculated lower-bound of the absorbed conjunction (disjunction) of premises.*

One may view the absorption relation $\Vdash_A$ as a restricted consequence relation that enjoys the following properties.

PROPOSITION 3. *The following properties hold for* $\Vdash_A$.

– $\langle \phi, e \rangle \Vdash_A \langle \phi, e \rangle$
[Reflexivity],

– *If* $\langle \phi', e'_1 \rangle \circ \langle \phi'', e''_1 \rangle \equiv \langle \psi', e'_2 \rangle \circ \langle \psi'', e''_2 \rangle$ *and* $\langle \phi', e'_1 \rangle \circ \langle \phi'', e''_1 \rangle \Vdash_A \langle \gamma, e_3 \rangle$, *then* $\langle \psi', e'_2 \rangle \circ \langle \psi'', e''_2 \rangle \Vdash_A \langle \gamma, e_3 \rangle$, *where* $\phi' \circ \phi'' \equiv \gamma \equiv \psi' \circ \psi''$.
[Left Logical Equivalence],

– *If* $\langle \psi, e_2 \rangle \models_M \langle \gamma, e_3 \rangle$ *and* $\langle \phi', e'_1 \rangle \circ \langle \phi'', e''_1 \rangle \Vdash_A \langle \psi, e_2 \rangle$ *then* $\langle \phi', e'_1 \rangle \circ \langle \phi'', e''_1 \rangle \models_M \langle \gamma, e_3 \rangle$
[Restricted Right Weakening].

Proof. Reflexivity follows by Corollary 1(i) and *Left Logical Equivalence* follows from Corollary 1(ii).

For *Restricted Right Weakening*, suppose the antecedent holds. By Theorem 3, $M \models \langle \phi', e'_1 \rangle \circ \langle \phi'', e''_1 \rangle$ only if $M \models \langle \psi, e_2 \rangle$ only if $[\![\phi']\!]_M \circ [\![\phi'']\!]_M \supseteq [\![\psi]\!]_M$. By Corollary 1, $e_2 \geq e_3$ and $[\![\psi]\!]_M \supseteq [\![\gamma]\!]_M$. So $\langle \phi', e'_1 \rangle \circ \langle \phi'', e''_1 \rangle \models_M \langle \gamma, e_3 \rangle$, by Proposition 2 (weak monotonicity).

Our absorption rules fail to satisfy four well-known properties, namely *Right Weakening*, *Or*, *And* and *Cautious Monotonicity*, displayed below in terms of a relation $\Vdash$ defined on propositional formulas.

– $\dfrac{\models_M \psi \rightarrow \gamma; \quad \phi \Vdash \psi}{\phi \Vdash \gamma}$ **[Right Weakening]**

– $\dfrac{\gamma \Vdash \phi; \quad \gamma \Vdash \psi}{\gamma \Vdash \ulcorner \phi \wedge \psi \urcorner}$ **[And]**

– $\dfrac{\phi \Vdash \gamma; \quad \psi \Vdash \gamma}{\ulcorner \phi \vee \psi \urcorner \Vdash \gamma}$ **[Or]**

– $\dfrac{\phi \Vdash \psi; \quad \psi \Vdash \gamma}{\ulcorner \phi \wedge \psi \urcorner \Vdash \gamma}$ **[Cautious Monotonicity]**

The main difference between *Right Weakening* and *Restricted Right Weakening* is the occurrence of $\models_M$ in our restricted version in place of $\Vdash_A$ in the conclusion position. Notice that this change is necessary because the *weak monotonicity property* for calculated bounds holds for $\models_M$ but does not hold for absorption: DA and CA are rules for calculating the bounds of a complex bounded formula *in its entirety.* Since γ can be a sub-formula of ψ, then e_3 may be less than e_2, by

Corollary 1. In such cases $\langle \gamma, e_3 \rangle$ *could not* be the absorption of both $\langle \phi', e_1' \rangle \circ \langle \phi'', e_1'' \rangle$ and $\langle \psi, e_2 \rangle$.

The remaining rules fail for absorption on syntactic grounds, which may be seen immediately if we substitute bounded formulas for propositional formulas. Only absorbed formulas appear in the conclusion position of CA and DA, which is incompatible with *And*, while CA and DA are configured only to apply to purely conjunctive or purely disjunctive complex bounded formulas, which is incompatible with the generality of both *Or* and *Cautious Monotonicity.*

The reason that we mention that absorption fails to satisfy these four properties is that most probabilistic logics are constructed around the axiom System P (Kraus, Lehmann and Magidor 1990) which consists of *Reflexivity*, *Left Logical Equivalence*, *Right Weakening*, *And*, *Or*, and *Cautious Monotonicity.* These remarks suggest that a logic for rational acceptance constructed around CA and DA will be fundamentally different from most probabilistic logics. We see that CA itself translates to a restricted version of the conjunction principle of System P, and cautious monotonicity is not satisfied as a consequence. Thus, System Y marks a distinction between logics for rational acceptance and logics for reasoning *about* probabilities (Halpern 2003). Furthermore, the approach under development differs from Adams's system (Adams 1975), Pearl's System Z (Pearl 1990), the epistemic probabilistic logic of Fagin, Halpern and Megiddo (Fagin, *et. al.* 1990) and epistemic probabilistic logics of (Halpern 2003), and probabilistic defaults of (Lukasiewicz 2002).

Viewing the relationship between probability and logic in this manner is motivated by observing the logical structure of statistical reasoning (Kyburg, Teng and Wheeler, forthcoming). Standard inferential statistical reasoning provides persuasive examples of reasonable, nonmonotonic inference forms (Kyburg and Teng 1999). It was remarked in (Wheeler 2004), where the notion of a bounded formula is introduced, that a default logic designed for this knowledge representation task should not be expected to satisfy the axioms of System P. The lottery paradox may be viewed then as a compact demonstration of the conflict resulting from adopting a system to model rational acceptance whose axioms satisfy System P.

3. Applying CA and DA

The expressive features of bounded formulas and the rules CA and DA allow us to observe a distinction that is important for resolving the paradoxes of rational acceptance. The distinction System Y allows us to observe is the difference between the probability of a conjunctive (disjunctive) event and the conjunction (disjunction) of a collection of probabilistic events.

Call a conjunction (disjunction) of basic bounded formulas an **out** expression and the inner measure of a conjunction (disjunction) an **in** expression. (An **in** expression is simply a bounded formula, whereas an **out** expression is a non-nested or "flat" complex bounded formula.) Thus, the rules CA and DA intuitively tell us how to convert an **out** expression to an **in** expression. Theorem 3 specifies how we may pass from specific Boolean combinations of basic bounded formulas to corresponding **in** expressions.

A consequence of this notation is that **out** expressions are necessarily of depth 1. However, if we wish to absorb a *nested* complex bounded formula, we may do so by applying DA and CA iteratively on the structure of the complex bounded formula. In this way we may effectively recover *And* and *Or*. To illustrate, consider the following example.

EXAMPLE 1. *The expression*

$$(\langle p, e_1\rangle \vee \langle q, e_2\rangle) \wedge \langle r \vee \neg s, e_3\rangle$$

is depth 2. Thus, the expression may be absorbed in two steps: First apply DA to the ***out*** *expression*

$$\langle p, e_1\rangle \vee \langle q, e_2\rangle$$

to yield the ***in*** *expression* $\langle p \vee q, e'_A\rangle$*, where* $e'_A = \max(e_1, e_2)$*, and then CA to the* ***out*** *expression*

$$\langle p \vee q, e'_A\rangle \wedge \langle r \vee \neg s, e_3\rangle$$

to yield the desired ***in*** *expression* $\langle (p \vee q) \wedge (r \vee \neg s), e''_A\rangle$*,where* $e''_A = \max(0, (e'_A + e_3) - 1)$.

Informally, an **in** expression says that we have specific information regarding the lower probability for a proposition or event. An **out** expression says that we have a collection of **in** expressions, each with a specific lower-bound probability. We may interpret a conjunction or disjunction of **in** expressions as either an intention to effect the appropriate absorption or a conjecture that the corresponding absorption is sound with respect to some threshold point for acceptance. However, as remarked above, a conjunction (disjunction) of **in** expressions isn't very informative. This observation motivates the design of the rules CA and DA and the notion of an **out** expression. The paradoxes of rational acceptance arise from stories that invite us to collapse this distinction between **out** and **in** expressions.

The interesting question raised by viewing the paradoxes in this light is whether one can define a logic that specifies when, and under what conditions, we may move from an **out** expression to an **in** expression.

In the problem of restricted entailment for rational acceptance, the rules CA and DA permit any **out** expression to be combined to make an **in** expression while preserving the *glb*. However, not all absorbed formula will be of interest: only those with sufficiently high probability will be candidates for acceptance. Given a threshold point for acceptance θ, the left coordinate of an **in** expression is accepted just in case the right coordinate is greater than or equal to θ. Hence, with respect to a threshold point for acceptance θ, a proposition p is accepted if and only if $\langle p, e\rangle$ holds and $e \geq \theta$.

I now return to the lottery paradox and the paradox of the preface to describe how this proposal works.

EXAMPLE 2 (Lottery Paradox).

The setup for the lottery—a fair 1000 ticket lottery and a threshold point for rational acceptance of $\theta = 0.99$*—means that when we consider each ticket* $T_1, ..., T_{1000}$ *individually, it is rational to accept that each ticket will lose. Suppose* $\neg T_i$ *represents that the ith ticket loses. The lottery setup gives us a set of bounded formulas*

$$L = \{\langle \neg T_1, 0.999\rangle, ..., \langle \neg T_{1000}, 0.999\rangle\}$$

and the problem comes when we are asked to interpret this collection. The question of how to interpret various combinations of losing tickets reduces to the question of what applications of DA or CA may be applied to $\langle \neg T_1, 0.999\rangle \ldots \langle \neg T_k, 0.999\rangle$*, for* $1 \leq k \leq 1000$ *where* $e_A \geq \theta$*.*

For instance, $\langle \neg T_i, 0.999\rangle \wedge \langle \neg T_j, 0.999, \rangle$ *may be absorbed, for any* i, j*, since* $\max(0, 0.999 + 0.999 - 1) = 0.998 \geq \theta$*. Hence all tickets may be pairwise conjoined: for all* $i \neq j : 1 \leq i \leq j \leq 1000$*, applying CA to any* ***out*** *expression of form* $\langle T_i, 0.999\rangle \wedge \langle T_j, 0.999\rangle$ *yields the* ***in*** *expression* $\langle \neg T_i \wedge \neg T_j, 0.998\rangle$*.*

Of course, we're only interested in those absorbed ***in*** *expressions where* e_A *is above threshold. Since* $0.998 > \theta$*, we know that the absorption of any pairing in the set* L *is each above the threshold point* θ *for acceptance. We can see that using CA we may determine each* ***in*** *expression of conjoined tickets both consistent and above the threshold point* θ *for acceptance.*

One point to notice about this construction is that it is applicable in cases where we may not know for sure that at most one ticket wins.

Thus, CA does not presuppose the weak lottery assumption necessary to the proposal in (Hawthorne and Bovens 1999) to calculate lower bound probability. For discussion see (Wheeler 2005).

EXAMPLE 3 (The Paradox of the Preface).

The setup for the preface paradox is that an author rationally accepts each page of an n page manuscript is without a mistake, but holds that it is rational to accept that there is at least one error appearing in the n page manuscript. Suppose that $\neg E_i$ *represents that there is no error on page* i *of the manuscript, and let* $\langle \neg E_i, e_i \rangle$ *represent that the lower bound on the probability that* $\neg E_i$ *is* e_i*. Suppose a threshold point of acceptance of* θ *such that*

$$\langle \neg E_1, e_1 \rangle \wedge \ldots \wedge \langle \neg E_n, e_n \rangle$$

and for all $i : 1 \leq i \leq n$, $\langle \neg E_i, e_i \rangle$ $e_i \geq \theta$ *but* $\langle E_1 \vee ... \vee E_n, e' \rangle$ *and* $e' \geq \theta$.

The twist to this example is the inclusion of the ***in*** *expression representing the preface statement, namely*

$$\langle E_1 \vee ... \vee E_n, e' \rangle,$$

where $e' \geq \theta$*. But, again, it is important not to interpret the Boolean combination of bounded formulas as entailing the corresponding absorbed combination. In other words,*

$$\langle \neg E_1, e_1 \rangle \wedge \ldots \wedge \langle \neg E_n, e_n \rangle \wedge \langle E_1 \vee \ldots \vee E_n, e' \rangle \quad (1)$$

is not necessarily inconsistent.

Incoherence would arise if we used unrestricted ***out*** *expressions to carry out successive disjunction eliminations on the bounded proposition* $E_1 \vee \ldots \vee E_n$ *within (1) until a contradiction was derived. But, before each of these deductive steps can be performed (within the scope of the bound of some* ***in*** *expression), there is an* ***out*** *expression composed of the preface apology and one of the statements reporting that a specific page of the book is error-free. There will be an application of CA of the following form: For some* $j \leq n$*, successive applications of CA to the expression*

$$\langle \neg E_1, e_1 \rangle \wedge \ldots \wedge \langle \neg E_j, e_j \rangle \wedge \langle E_1 \vee ... \vee E_n, e' \rangle \quad (2)$$

will yield the ***in*** *expression*

$$\langle (\neg E_1 \wedge \ldots \wedge \neg E_j) \wedge (E_1 \vee ... \vee E_n), e_A \rangle \quad (3)$$

where $e_A = \max(0, (e_1 + \cdots + e_j + e') - n)$*. We may, then, resolve the clause in (3). Since we are restricting ourselves to looking at only those* *absorbed conjunctions that are above threshold and, by hypothesis, (1) is coherent, $j < n$. Hence, a procedure that applied CA to expressions of form (2) would block absorption before the resulting contradictory* ***in*** *clause could be resolved.*

Before concluding, I would like to draw attention to a feature of this proposal that I think is a positive consequence of the type of approach—the structural view—that I am advocating. The rules CA and DA are designed to mark, syntactically, the source of the problem for rational acceptance and entailment, what I've called here *absorption*. Notice that my proposed solution does not hinge on artificial features of the lottery paradox or paradox of the preface thought experiments, but rather attempts to address the conflict in the principles of rational acceptance that these paradoxes and their variants exhibit. This framework is applicable if there is no guaranteed winning ticket but only a likely winner, which was noted in comparison to the example just above; the framework is applicable to biased lotteries; and it is applicable if there is more than one accepted but false proposition (e.g., more than one winning ticket.). So long as you have a threshold point for acceptance and you know the lower-bound acceptance point for each of the propositions that you've accepted, then you may consider arguments formed from collections of so-accepted statements by interpreting their logical combination and resolving the consequences within the absorbed **in** expressions.

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