Pumping Lemma - informal

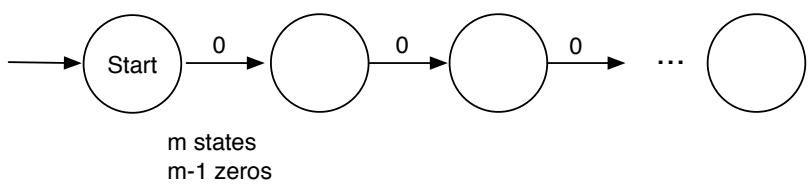
Imagine somebody proposes a finite state machine for $0^n 1^n$, which is the concatenation of a string of *n* zeros followed by a string of *n* ones.

Since it is **FINITE**, we can ask how many states that machine has. (Pick a number:)

32

Great. Now, here is a string that you should be able to compute: $0^{32}1^{32}$.

Note: The first segment needs to verify 32 zeros. Since the start state takes up one state, you have at most 31 individual new states to generate zeros and that is not enough. So, somewhere you will have to hit a state twice. And this generalizes:



Okay. So I say to this person: I KNOW there is a loop, so tell me the first state that gets entered twice. More specifically, tell me these three things about your machine:

- Tell me how many zeros are read before the loop
- Tell me how many zeros are in the loop
- How many zeros and 1s after the loop.

Pick any numbers you like. For example,

$$0^{27}0^{3}0^{2}1^{32}$$

where

- x = 27 (initial segment of 0s)
- y = 3 (loop of 0s)
- z = 2 (remaining 0s after the loop) + 32 (1s after the loop)

Once these parameters are pinned down, NOW try to run a different string on the machine:

$$0^{27}0^{3}0^{3}0^{2}1^{32}$$

1. This reads 27 0s.
2. Then loops for 3 0s. [*Just like above.*]
3. Then it sees the exact same thing; it will be in exactly the same place as at the end of step (2), but with 3 more 0s. [*The Pumping Step*]
4. Continues as before adding 2 0s then 32 1s.
   - But this machine accepts $0^{35}1^{32}$.

What you are doing here is PUMPING UP the set. Call this property P.

If a set is accepted by a Finite State Machine, the set has property P.

$$Regular\ Set \rightarrow P$$

In other words, if a string is accepted by an FSM, then pumped up versions of that string are accepted, too. Pumping up a regular set yields a regular set.

But, we are interested in this form:

$$\neg P \rightarrow \neg Regular\ Set$$

In other words, if we can show that a set DOES NOT satisfy the pumping property, then we know the set is not regular.

> *Remark*: Regular sets are called regular because if you have a regular set, you can always pump it up at regular (linear) intervals and get other things in the set.

This is why anything that grows faster than linear is NEVER regular.

Pumping Lemma

If A is a regular set,
then:
$\exists n.\ n$ is the number of states in the machine for A, **and**
$\forall s \in A, where\ |s| \geq n,$
$\exists x, y, z$ such that
i) $s = xyz$ **and** [s is a string in regular set A]
$ii)$ $|xy| \leq n$ **and** [loop comes in first n symbols:
$iii)$ $|y| \geq 1$ **and** x is initial segment;
y is first non-empty loop]

$iv)$ $\forall i \geq 0, xy^i z \in A.$ [pumping the loop y up with i states yields a string also in A.]

> *Remark*: when $i = 0$, this means that the state y is "popped": the loop in y is taken out. When your pumping number is 0, you are cutting the string down from the original by cutting the loop out. Note that sometimes this case is what is needed to show that a string is not regular.

Negation of the Pumping Property:

~P:

$\forall n. n$ is the number of states in any machine for S,
$\exists s \in A, where\ |s| \geq n$, such that
$\forall x, y, z$ where $z = vwx$

i) $s = xyz$ **and** [*same as before*]
$ii)$ $|xy| \leq n$ **and**
$iii)$ $|y| \geq 1$ **and**

$iv)$ $\exists i \geq 0, xw^i x \notin S$ [pumping the loop *y* up with *i* states yields a string that ***IS NOT*** in A.]

EXAMPLE 16. The set of palindromes are not regular.

1) Suppose there is a machine that has at least *k* states.

$$s = 0^k 1 0^k$$

2) Now consider some way to slip *z* into *xyz*:

$$0^k 1 0^k = xyz,$$

where note that $y = 0^m$ for some $1 \leq m \leq k$.

Remark: no matter how you split up the segment *xyz* into 3 parts, I know that *xy* is the 0^k segment and I know that *y* will be all 0s.

3) Now let $i = 2$ and fill in the last clause of the lemma:

$$xy^2z = 0^{k+m} 1 0^k$$

Since $m \geq 1$, then $0^{k+m} 1 0^k \neq 0^k 1 0^{k+m}$

So, the pumping property does not hold for palindromes. Therefore, the set of palindromes is not regular. Q.E.D.